HW **Push-outs and Pull-backs.**

Proposition. Let P be an R-module. The following conditions are equivalent:

(1) P is projective.
(2) $\mathrm{Hom}_R(P, \_)$ is an exact functor.
(3) Every epimorphism $\beta\colon M \to P$ splits.
(4) P is isomorphic to a summand of a free R-module.

Definition. A ring is called **left hereditary** if all left ideals are projective. A hereditary integral domain is called a **Dedekind domain**.

Theorem. (1) If R is a left hereditary ring then every submodule of a projective left R-module is projective.
(2) If R is a Dedekind domain then every **finitely generated** torsion free R-module is isomorphic to a direct sum of ideals.
Furthermore, if $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ are ideals in a Dedekind domain R then $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m \approx \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ if and only if $m = n$ and the ideals $\mathfrak{a}_1 \cdots \mathfrak{a}_m$ and $\mathfrak{b}_1 \cdots \mathfrak{b}_n$ are isomorphic.

Proposition. If P is a projective R-module and I a (two-sided) ideal, then P/IP is a projective R/I-module.

Proposition. Let P_1 and P_2 be projective R-modules and let J be the Jacobson radical of R. If $P_1/JP_1 \approx P_2/JP_2$ then $P_1 \approx P_2$.

Proposition. If R is a local ring or a PID then all projective modules are free.

Corollary. A **finitely generated** module M over a commutative noetherian ring R is projective if and only if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$.

Proposition. If an indecomposable projective module M has finite length then there exists a unique maximal proper submodule M_0 of M. Furthermore, M/M_0 is a simple module and M is determined up to isomorphism by M/M_0.

Proposition. Let R be an integral domain and K its quotient field. Let M be an R-submodule of K. Then M is projective if and only if there is an R-submodule P of K such that $MP = R$.

Proposition. Let E be an R-module. The following conditions are equivalent:

(1) E is injective.
(2) $\mathrm{Hom}_R(\_, E)$ is an exact functor.
(3) Every monomorphism $\alpha\colon E \to M$ splits.

Definition. Divisible module.

Proposition An injective module over an integral domain is divisible.

Corollary. An injective module over an integral domain is faithful.

Proposition. If R is an integral domain with quotient field K, then K is an injective R-module.

Baer's Theorem. A left R-module E is injective if and only if every homomorphism φ from a left ideal L of R into E extends to a homomorphism from R into E.

HW **Proposition.** (1) A direct product of injective modules is injective.
(2) If R is noetherian, then a direct sum of injective modules is injective.

Proposition. An injective module over a noetherian (not necessarily commutative) ring R is a (possibly infinite) direct sum of indecomposable injective modules.

Corollary. If R is a PID then a module M is injective if and only if it is divisible.

Theorem. A ring R is semi-simple if and only if every R-module is projective [injective].

Essential Extensions and Injective Envelopes

Propostion. Let N be a submodule of M. The following are equivalent:

(1) Every $m \neq 0 \in M$ has a non-trivial multiple which belongs to N.
(2) Every non-trivial submodule of M has a non-trivial intersection with N.
(3) Whenever $\varphi\colon M \to X$ is a homomorphism and the restriction of φ to N is monic then φ is monic.

Definition. In this case we say that N is an **essential submodule** of M or that M is an **essential extension** of N.

Example. If S is a multiplicative set in the center of R and M an R-module and $\theta\colon M \to S^{-1}M$ is the canonical map, then $S^{-1}M$ is an essential extension of $\theta(M)$.

Proposition. If N is an injective module and M is an essential extension of N, then $M = N$.

Corollary. If M is an injective module over a commutative noetherian ring R and $S = R \smallsetminus \bigcup \operatorname{Ass} M$, then $S^{-1}M \approx M$.

Theorem. Let E' be an injective module and $M \subseteq E \subseteq E'$. The following conditions are equivalent:

(1) E is a maximal essential extension of M.
(2) E is a minimal injective submodule of E' containing M.
(3) E is an essential extension of M and is injective.
(4) E is injective and every **monomorphism** from M into an injective module Q extends to a **monomorphism** from E into Q.

Definition. Injective envelope.

Proposition. If N is a submodule of an injective module E then there exists an injective envelope of N contained in E.

Theorem. Existence of injective envelopes.

HW **Proposition.** Let E be an injective module. The following conditions are equivalent:

(1) E is indecomposable.
(2) E is the injective envelope of each of its non-trivial submodules.
(3) Every two non-trivial submodules of E intersect non-trivially.

HW **Corollary.** If E is an indecomposable injective module over a commutative noetherian ring R, then $\operatorname{Ass} E = \{\mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, E is isomorphic to the injective envelope of $R/\mathfrak{p}$, and $E \approx E_{\mathfrak{p}}$.

Example. The indecomposable injective $\mathbb{Z}$-modules.

HW **Proposition.** Let E be an indecomposable injective module.

(1) Every monic endomorphism of E is in fact an automorphism.
(2) $\operatorname{End}_R E$ is a local ring.

Theorem. If R is a (not necessarily commutative) artinian ring, then every injective R-module is a direct sum of indecomposable injective R-modules. Furthermore every indecomposable injective module is the injective envelope of a unique simple module. **In most situations, at least,** an indecomposable module over an artinian ring has finite length.

Lemma. Let R be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and let E be the injective envelope of $R/\mathfrak{m}$. If M is any module with finite length, then $\operatorname{Hom}_R(M, E)$ also has finite length, in fact $\operatorname{length} \operatorname{Hom}_R(M, E) = \operatorname{length} M$.

Proposition. A finitely generated injective module M over a commutative noetherian ring has finite length.

PROOF: It suffices to show that every associated prime for M is maximal. Now if $\mathfrak{p} \in \operatorname{Ass} M$ then M contains a summand isomorphic to $E(R/\mathfrak{p})$ and thus $E = E(R/\mathfrak{p})$ is finitely generated. Now if $\mathfrak{p} \subsetneq \mathfrak{m}$ then $\mathfrak{m}E = E$. Furthermore, $E \approx E_{\mathfrak{m}}$, so we may replace R by $R_{\mathfrak{m}}$. Then by NAK we conclude that $E = 0$, a CONTRADICTION. Thus $\mathfrak{p}$ must be maximal. ☑

Proposition. If R is a commutative local ring with maximal ideal $\mathfrak{m}$ then R is artinian if and only if there exists a non-trivial finitely generated injective R-module. In this case, the injective envelope of $R/\mathfrak{m}$ is finitely generated.

PROOF: Let E be the injective envelope of $R/\mathfrak{m}$. Since $E \approx \mathrm{Hom}_R(R, E)$, it follows from the lemma above that E has finite length if and only if R does (which is equivalent to R being artinian.)

On the other hand, if there exists any finitely generated injective module M then we have seen that M has finite length. Hence $\mathrm{Ass}\, M = \{\mathfrak{m}\}$ and so M contains a copy of $R/\mathfrak{m}$. Then M contains a copy of the injective envelope E of $R/\mathfrak{m}$, so E has finite length.