

## TENSOR PRODUCTS

### Balanced Maps.

**Note.** One can think of a balanced map  $\beta : L \times M \rightarrow G$  as a multiplication taking its values in  $G$ . If instead of  $\beta(\ell, m)$  we write simply  $\ell m$  (a notation which is often undesirable) the rules simply state that this multiplication should be distributive and associative with respect to the scalar multiplication on the modules. I. e.

$$\begin{aligned}(\ell_1 + \ell_2)m &= \ell_1 m + \ell_2 m \\ \ell(m_1 + m_2) &= \ell m_1 + \ell m_2, \quad \text{and} \\ (\ell r)m &= \ell(rm).\end{aligned}$$

There are a number of obvious examples of balanced maps.

(1) The ring multiplication itself is a balanced map  $R \times R \rightarrow R$ .

(2) The scalar multiplication on a left  $R$ -module is a balanced map  $R \times M \rightarrow M$ .

(3) If  $K$ ,  $M$ , and  $N$  are left  $R$ -modules and  $E = \text{End}_R M$  then for  $\delta \in \text{End}_R M$  and  $\varphi \in \text{Hom}_R(M, P)$  the product  $\varphi\delta$  is defined in the obvious way.

This gives  $\text{Hom}_R(M, P)$  a natural structure as a right  $E$ -module. Analogously,  $\text{Hom}_R(K, M)$  is a left  $E$ -module in an obvious way. There is an  $E$ -balanced map  $\beta : \text{Hom}_R(M, P) \times \text{Hom}_R(K, M) \rightarrow \text{Hom}_R(K, M)$  given by  $(\psi, \varphi) \mapsto \psi\varphi$ .

(4) If  $M$  and  $P$  are left  $R$ -modules and  $E = \text{End}_R M$ , then, as above,  $\text{Hom}_R(M, P)$  is a right  $E$ -module. Furthermore,  $M$  is a left  $E$ -module in an obvious way. We then get an  $E$ -balanced map

$$\text{Hom}_R(M, P) \times M \rightarrow P$$

by  $(\varphi, m) \mapsto \varphi(m)$ .

(5) A (distributive) multiplication on an abelian group  $G$  is a  $\mathbb{Z}$ -balanced map  $G \times G \rightarrow G$ .

It is interesting to note that in examples (1) through (4), the target space for the balanced map is in fact an  $R$ -module, rather than merely an abelian group. This is not untypical.

### Universal Characterization of Tensor Product.

$$\begin{array}{ccc} L \times M & & \\ \theta \downarrow & & \\ L \otimes_R M & \longrightarrow & G. \end{array}$$

For  $\ell \in L$  and  $m \in M$  we let  $\ell \otimes m$  denote the element  $\theta(\ell, m) \in L \otimes_R M$ .

**Proposition.** The elements  $\ell \otimes m$  **generate**  $L \otimes_R M$ . (In general, however, not every element of  $L \otimes_R M$  has the form  $\ell \otimes m$ .)

**Proposition.** (1)  $\text{Hom}_R(R, M) \approx M$

(2)  $R \otimes_R M \approx M$ .

**Proposition.** (1)  $R/I \otimes_R M \approx M/IM$ .

(2)  $S^{-1}R \otimes_R M \approx S^{-1}M$ .

**HW**

(3) If  $\mathfrak{a}$  and  $\mathfrak{b}$  are ideals in a commutative ring  $R$  then  $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \approx R/(\mathfrak{a} + \mathfrak{b})$ .  
(In particular,  $R/\mathfrak{a} \otimes_R R/\mathfrak{b} = 0$  if  $\mathfrak{a}$  and  $\mathfrak{b}$  are co-maximal.)

Note that the above allows one to compute  $L \otimes_R M$  explicitly when  $R = \mathbb{Z}$  and  $L$  and  $M$  are **finitely generated** abelian groups.

**Proposition.** If  $\varphi: L \rightarrow L'$  and  $\psi: M \rightarrow M'$  then there exists a unique map  $\varphi \otimes_R \psi: L \otimes_R M \rightarrow L' \otimes_R M'$  making the diagram below commute.

$$\begin{array}{ccc} L \times N & \xrightarrow{\varphi \times \psi} & L' \times M' \\ \theta \downarrow & & \theta' \downarrow \\ L \otimes_R M & \xrightarrow{\varphi \otimes_R \psi} & L' \otimes_R M'. \end{array}$$

**HW**

**Proposition.** Let  $R$  be an integral domain. If  $L$  is a divisible  $R$ -module and  $M$  is torsion, then  $L \otimes_R M = 0$ .

**Corollary.** If  $G$  is a torsion divisible abelian group (for instance  $\mathbb{Q}/\mathbb{Z}$ ) then there is no (unitary) ring whose additive group is isomorphic to  $G$ . In fact, the only multiplication on  $G$  is the trivial one.

**Proposition.** If  $K$  is a commutative ring and  $R$  is a  $K$ -algebra, then  $L \otimes_R M$  is a  $K$ -module. In particular, if  $R$  is commutative then  $L \otimes_R M$  is an  $R$ -module.

**Proposition.** If  $L$  and  $M$  are finitely generated modules over a commutative ring  $R$  then  $L \otimes_R M$  is a finitely generated  $R$ -module.

**Tensor Products of Rings.** The tensor product as coproduct in the category of commutative rings.

**Proposition.** If  $R$  is a subring of a ring  $S$  or, more generally, there is a ring morphism  $\rho: R \rightarrow S$ , then for every left  $R$ -module  $M$ ,  $S \otimes_R M$  is a left  $S$ -module. The resulting functor  $M \mapsto S \otimes_R M$  is left adjoint to the forgetful functor from  $S$ -modules to  $R$ -modules.

$$\begin{array}{ccc} M & & \\ \downarrow & & \text{Hom}_R(M, P) \approx \text{Hom}_S(S \otimes_R M, P). \\ S \otimes_R M & \longrightarrow & P \end{array}$$

**Example.** The complexification of a real vector space.

**Motivational Example.** Let  $W$ ,  $X$ , and  $Y$  be **sets** and let  $W^Y$  denote the set of functions from  $Y$  into  $W$ . (This is standard notation in set theory.) If  $f: X \times Y \rightarrow W$  is any function, then for each  $x \in X$  there is a function  $f_x: Y \rightarrow W$  given by  $f_x(y) = f(x, y)$ . Thus there is a function  $\tilde{f}: X \rightarrow W^Y$  given by  $x \mapsto f_x$ . One easily sees that  $f \mapsto \tilde{f}$  is a natural (!) one-to-one correspondence between the set of functions  $X \times Y \rightarrow W$  and the set of functions  $X \rightarrow W^Y$ . We can indicate this schematically as follows:

$$\frac{X \times Y \rightarrow W}{X \rightarrow W^Y} \qquad \text{Hom}_{\text{Sets}}(X \times Y, W) \approx \text{Hom}_{\text{Sets}}(X, W^Y).$$

**Theorem.** The tensor product is left adjoint to Hom.

$$\frac{L \otimes_R M \rightarrow G \quad [\mathbb{Z}\text{-linear}]}{L \rightarrow \text{Hom}_{\mathbb{Z}}(M, G) \quad [R\text{-linear}]} \qquad \text{Hom}_{\mathbb{Z}}(L \otimes_R M, G) \approx \text{Hom}_R(L, \text{Hom}_{\mathbb{Z}}(M, G)).$$

(NOTE: For  $\varphi \in \text{Hom}_{\mathbb{Z}}(M, G)$  and  $r \in R$ ,  $\varphi r$  is defined by  $\varphi r(m) = \varphi(rm)$ .)

**Theorem.** There exist natural maps  $L \rightarrow \text{Hom}_{\mathbb{Z}}(M, L \otimes_R M)$  and  $\text{Hom}_{\mathbb{Z}}(M, G) \otimes_R M \rightarrow G$  where

$$\begin{aligned} \ell &\mapsto \hat{\ell} \in \text{Hom}_{\mathbb{Z}}(M, L \otimes_R M) \quad \text{with} \quad \hat{\ell}(m) = \ell \otimes m \\ \text{and} \quad \varphi \otimes m &\mapsto \varphi(m) \in G. \end{aligned}$$

PROOF: This is fairly obvious, but it is also enlightening to note that these maps can be obtained by applying the theorem above as follows:

$$\frac{L \otimes_R M \xrightarrow{\text{identity}} L \otimes_R M \quad [\mathbb{Z}\text{-linear}]}{L \rightarrow \text{Hom}_{\mathbb{Z}}(M, L \otimes_R M) \quad [R\text{-linear}]}$$

$$\frac{\text{Hom}_{\mathbb{Z}}(M, G) \otimes_R M \rightarrow G \quad [\mathbb{Z}\text{-linear}]}{\text{Hom}_{\mathbb{Z}}(M, G) \xrightarrow{\text{identity}} \text{Hom}_{\mathbb{Z}}(M, G) \quad [R\text{-linear}]}$$

**Proposition.** The tensor product is right exact and commutes with arbitrary direct sums.

**Proposition.** If  $L$  and  $M$  are free modules over a commutative ring  $R$  then  $L \otimes_R M$  is a free  $R$ -module.

**Proposition.** The tensor product of vector spaces.

**Example.** The tensor product is not usually left exact.

**Proposition.**  $\text{Hom}_{\mathbb{Z}}(R, \_)$  is a functor taking abelian groups to left  $R$ -modules. (For  $\varphi \in \text{Hom}_{\mathbb{Z}}(R, G)$  and  $r \in R$ ,  $r\varphi$  is defined by  $r\varphi(r') = \varphi(r'r)$ .) It is right adjoint to the forgetful functor from  $R$ -modules to  $\mathbb{Z}$ -modules.

$$\text{Hom}_{\mathbb{Z}}(M, G) \approx \text{Hom}_{\mathbb{Z}}(R \otimes_R M, G) \approx \text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, G)).$$

**Proposition.** If  $G$  is an injective  $\mathbb{Z}$ -module then  $\text{Hom}_{\mathbb{Z}}(R, G)$  is an injective  $R$ -module.

**Proposition.** If  $M$  is an  $R$ -module and  $G$  a  $\mathbb{Z}$ -module and  $\alpha: M \rightarrow G$  a  $\mathbb{Z}$ -linear monomorphism, then the induced  $R$ -linear map  $\tilde{\alpha}: M \rightarrow \text{Hom}_{\mathbb{Z}}(R, G)$  is monic.

**Corollary.** Every module over an arbitrary ring  $R$  has an injective envelope.

**Definition.** A left  $R$ -module  $M$  is **flat** if whenever  $\alpha: L \rightarrow L'$  is a monomorphism of right  $R$ -modules then the induced map  $\alpha \otimes_R 1_M: L \otimes_R M \rightarrow L' \otimes_R M$  is also monic.

**Proposition.** a left  $R$ -module is flat if and only if whenever  $I$  is a left ideal of  $R$  and  $\alpha: I \hookrightarrow R$  is the inclusion map, then the map  $\alpha': I \otimes_R M \rightarrow M$  given by  $i \otimes m \mapsto im$  is a monomorphism. (NOTE: It suffices to check this for finitely generated left ideals.) (NOTE: This is analogous to Baer's criterion for injectivity of modules.)

**Proposition.** (1) Arbitrary direct sums of flat modules are flat.

(2) Direct summands of flat modules are flat.

(3) Projective modules are flat. In particular, the ring  $R$  itself is a flat left  $R$ -module.

(4)  $S^{-1}R$  is flat as both a left and right  $R$ -module.

**Proposition.** Flat modules over an integral domain are torsion free.

**Proposition.** If every finitely generated submodule of a module is flat then the whole module is flat.

WARNING: The converse is not usually true. A submodule of a flat module is usually not flat.

**Proposition.** A module over a Dedekind domain is torsion free if and only if it is flat. (NOTE: This characterizes Dedekind domains among noetherian integral domains.)

**Proposition.** A ring is von Neumann regular if and only if all submodules of flat modules are flat. (It suffices to check that all finitely generated left ideals are flat.)

**General Comments.** The construction of the tensor product is quite horrendous, and one's first thought is that one could never actually compute it. In fact, on the whole this is pretty much correct, despite the existence of specific cases where the computation is straightforward.

In fact, though, it is precisely these simple cases which are on the whole most useful, especially the case  $(R/I) \otimes_R M$ , where  $I$  is a (two-sided) ideal. It may seem pretentious to write  $(R/I) \otimes_R M$  instead of simply  $M/IM$ , but it is actually sometimes more useful or convenient to do so.

The other wide-spread use of the tensor product is for extension of scalars. Given a module  $M$  over a commutative ring  $R$  and an  $R$ -algebra  $R'$ , one gets the  $R'$ -module  $R' \otimes_R M$ . This construction has classically been widely used in the case of the "complexification" of a real vector space. (The classical construction was as follows: "Choose a basis for the real vector space  $V$ . Now form the complex vector

space  $V_{\mathbb{C}}$  on the same basis and identify  $V$  as a real subspace of  $V_{\mathbb{C}}$  by ... ” One then needed to show that this construction is independent of the choice of basis.)

There seem to be only a few cases where one is interested in the tensor product  $L \otimes_R M$ , where  $L$  is an arbitrary right  $R$ -module and  $M$  an arbitrary left  $R$ -module . However some of these cases are important.

The concept of flatness is very important, especially for modules over commutative rings. One could almost claim that it would have been worth inventing the tensor product simply in order to be able to define flat modules.

In general, one can wonder whether it even matters whether one can explicitly compute  $L \otimes_R M$  or not. The question is: Is  $L \otimes_R M$  important as an entity, or is it a merely a convenient way of talking about balanced maps (bilinear maps, in the case where  $R$  is commutative)?

The same point may be made in reference to  $\text{Hom}_R(M, N)$  and  $\text{End}_R M$ . In a lot of cases, we don't really care about these as entities or want to know their structure. They merely provide convenient ways of making statements about the existence of homomorphisms or endomorphisms.

For instance,  $\text{Hom}_R(M, N) = 0$  can be restated less concisely as “There are no non-trivial homomorphisms from  $M$  to  $N$ .”

The statement “The direct sum decompositions of  $M$  correspond to the idempotents in the ring  $\text{End}_R M$ ” can be restated as “There is a one-to-one correspondence between the direct sum decompositions of  $M$  and the idempotent endomorphisms of  $M$ .”

However there are some important cases where the constructions  $L \otimes_R M$  and  $\text{Hom}_R(M, N)$  are important ways of constructing new modules from existing ones. The most important of these is something called **Morita equivalence (duality)** which is a way of showing that the category of modules over one ring is isomorphic to the category of modules over a different ring.

For instance, if  $R$  is the ring of  $n \times n$  matrices over a field  $K$  then there is a Morita equivalence between the category of left  $R$ -modules and the category of vector spaces over  $K$ . (One can almost see this from the Wedderburn Theorem.)

## References On Multi-linearity, Symmetric Products, and Alternating Products.

- (1) Bourbaki, **Algebra**, Chapters 2 and 3.
- (2) Chevalley, **Fundamental Concepts of Algebra**, Academic Press (Pure & Applied Mathematics, vol. 7), 1956.