Finitely Generated p-Primary Modules over PIDs

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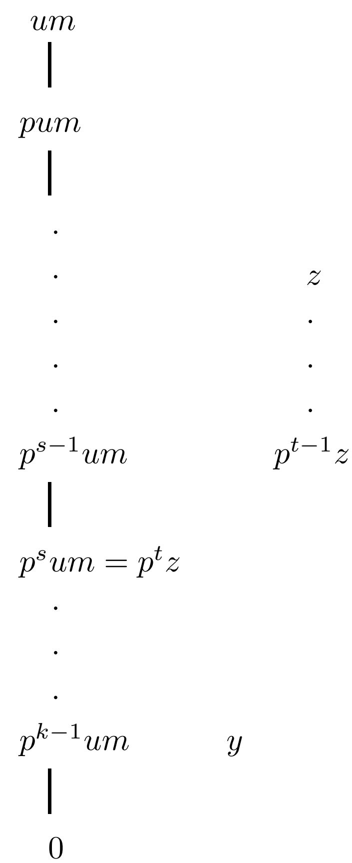
ASSUMPTIONS. R is a principal ideal domain and (p) is a prime ideal. M is a module such that $p^k M = 0$ and $p^{k-1}M \neq 0$. Furthermore $m \in M$ is such that $p^{k-1}m \neq 0$.

Claim. If $M \neq Rm$ then there exists $y \notin Rm$ such that $py = 0$.

PROOF: By assumption there exists $z \in M$ with $z \notin Rm$. Since $p^k z = 0 \in Rm$, there exists a smallest possible $t \geq 1$ such that $p^t z \in Rm$. (Why is $t \neq 0$?) Thus $p^t z$ is a multiple of m and we can write $p^t z = up^s m$ for $s \geq 0$ and $u \notin (p)$. (This uses the fact that R is a unique factorization domain.)

We claim that $s \neq 0$. Otherwise $p^t z = um$ with $u \notin p$ and $t \geq 1$ and $p^{k-t}um = p^{k-t}p^t z = p^k z \in p^k M = 0$, so $p^{k-t}u \in \operatorname{ann}(m) = (p^k)$, with $u \notin (p)$, and this contradicts unique factorization in R.

Thus $p^t z = p^s um$ with $s, t \geq 1$. Now let $y = p^{t-1}z - p^{s-1}um$. Then $py = 0$ and $y \notin Rm$, otherwise $p^{t-1}z = y + p^{s-1}um \in Rm$, contrary to the choice of t. This proves the claim.



Claim. The only submodules of Ry are 0 and Ry itself.

PROOF: Since $py = 0$, $(p) \subseteq \operatorname{ann} y$. Since (p) is a maximal ideal and $Ry \neq 0$, thus $\operatorname{ann} y = (p)$. Now Ry is the cyclic submodule generated by y, so $Ry \approx R/\operatorname{ann}(y) = R/(p)$. Thus the submodules of Ry are in one-to-one correspondence with the submodules (ideals) of R containing (p). But (p) is a maximal [proper] ideal, so the only ideals containing (p) are (p) itself and R, which correspond to the submodules 0 and Ry of Ry.

Definition. A non-trivial module S over a ring R is called **simple** if it has no submodules except for 0 and the module S itself.

Lemma. If $Rm \neq M$ then there exists $y \in M$ with $Ry \cap Rm = 0$.

PROOF: If y is chosen as above, then since $Ry \cap Rm$ is a submodule of Ry, and $Ry \cap Rm \neq Ry$ (since $y \notin Rm$), it follows from the **claim** above that $Ry \cap Rm = 0$.

Proposition. Let R be a PID and (p) be a prime ideal and M be a module such that $p^k M = 0$. Let $m \in M$ be such that $p^{k-1}m \neq 0$ and suppose that there exists a submodule $N \subseteq M$ which is maximal with respect to the property that $N \cap Rm = 0$. Then $M = N \oplus Rm$.

PROOF: Since $N \cap Rm = 0$ by construction, we need to show that $M = N + Rm$. Since there is a one-to-one correspondence between the submodules of M containing N and the submodules of M/N, it suffices to prove that $\dfrac{N + Rm}{N} = M/N$, which is the same as saying that in $R(m + N) = M/N$, i.e. that M/N is cyclic generated by the coset $m + N$.

Now look at M/N and let $x = m + N \in M/N$. Using the one-to-one correspondence between submodules of M/N and submodules of M containing N, we see that the condition that N is maximal w.r.t. the property $N \cap Rm = 0$ translates into the condition that

> 0 is a maximal submodule of M/N with respect to having trivial intersection with $R(m + N)$.

(This takes a little checking, since $N \not\subseteq Rm$.)
(This amounts to the assertion that $R(m + N)$ is an essential submodule of M/N.)
Notice also that $p^k(M/N) = 0$ and $p^{k-1}(m + N) \neq 0$ (since $p^{k-1}m \notin N \cap Rm$ because $N \cap Rm = 0$).

Now use the previous proposition to see that if $(N + Rm)/N \neq 0$ then there exists $y \in M/N$ such that $Ry \cap R(m + M) = 0$. But this contradicts the previous paragraph.

Thus $N + Rm = M$, as required. ☑