IN THIS COURSE ALL RINGS AND MODULES ARE UNITARY.
Furthermore, if R and S are rings and $\rho\colon R \to S$ is a ring morphism, it is required that $\rho(1) = 1$. Likewise if we say that R is a subring of S this means (among other requirements) that they have the same identity element.

1. **a)** Let $\rho\colon R \to R'$ be a morphism of commutative rings. Prove that if $\mathfrak{p}'$ is a prime ideal in R' then $\rho^{-1}(\mathfrak{p}')$ is prime in R.
b) Let R be a subring of R'. Prove that if $\mathfrak{p}'$ is a prime ideal in R' then $\mathfrak{p}' \cap R$ is a prime ideal in R.

Notation. If $\mathfrak{p}$ is a prime ideal in a commutative ring R and $S = R \setminus \mathfrak{p}$, then we write $M_{\mathfrak{p}} = S^{-1}M$.

Lemma [Hungerford, Theorem 2.2, p 378]. If S is a multiplicative set in a commutative ring R and $\mathfrak{a}$ is an ideal such that $\mathfrak{a} \cap S = \varnothing$ then there exists at least one ideal $\mathfrak{p}$ maximal with respect to the properties $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \cap S = \varnothing$. Furthermore, any such ideal is prime.

2. Let M be an R-module and $m \in M$ and let $\operatorname{ann} m = \{r \in R \mid rm = 0\}$. Prove that $m/1 \neq 0 \in S^{-1}M$ if and only if $S \cap \operatorname{ann} m = \varnothing$.

3. Let M, N, P be modules over a commutative ring R. Prove that:

(1) If $m_1, m_2 \in M$, then $m_1 = m_2$ if and only if $m_1/1 = m_2/1 \in M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.
(2) $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$.
(3) Suppose that $N, P \subseteq M$. Then $N = P$ if and only if $N_{\mathfrak{m}} = P_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.
(4) If $\varphi \in \operatorname{Hom}_R(M, N)$ then φ is a monomorphism [epimorphism] if and only if $\varphi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is monic [epic] for all maximal ideals $\mathfrak{m}$.

6. Recall the following result [Hungerford, Theorem 2.2, p 378]: If S is a multiplicative set in a commutative ring R and $\mathfrak{a}$ is an ideal such that $\mathfrak{a} \cap S = \varnothing$ then there exists at least one ideal $\mathfrak{p}$ maximal with respect to the properties $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \cap S = \varnothing$. Furthermore, any such ideal is prime.

($\Rightarrow$): Suppose that $\operatorname{Ass} M = \{\mathfrak{p}\}$. Then by the Lemma given in the homework, for every $m \neq 0 \in M$, $\operatorname{ann} m \subseteq \mathfrak{p}$ and every prime ideal containing $\operatorname{ann} m$ contains $\mathfrak{p}$. From this it follows first that if $s \notin \mathfrak{p}$ then $sm \neq 0$, showing that $m \notin \operatorname{Ker} \theta$, where $\theta\colon M \to M_{\mathfrak{p}}$ is the canonical map. Furthermore, let $r \neq 0 \in \mathfrak{p}$ and let $S = \{r^k \mid k \geq 1\}$. If $S \cap \operatorname{ann} m = \varnothing$ then by the above result from Hungerford there exists a prime ideal $\mathfrak{q}$ with $\mathfrak{q} \supseteq \operatorname{ann} m$ and $\mathfrak{q} \cap S = \varnothing$. But then $\mathfrak{q} \supseteq \mathfrak{p}$ and $r \notin \mathfrak{q}$, a CONTRADICTION. Thus there exists an element $r^k \in S \cap \operatorname{ann} m$, so $r^k m = 0$. Since $\mathfrak{p}$ is finitely generated (because R is noetherian) it then follows easily that $\mathfrak{p}^{k'} m = 0$ for some k'.

($\Leftarrow$): Now suppose the stated conditions hold and let $\mathfrak{q} \in \operatorname{Ass} M$. Then $\mathfrak{q} = \operatorname{ann} m$ for some m. By assumption, for some k, $\mathfrak{p}^k \subseteq \operatorname{ann} m = \mathfrak{q}$ Then $\mathfrak{p} \subseteq \mathfrak{q}$ since $\mathfrak{q}$ is prime. On the other hand, if $s \notin \mathfrak{p}$ then $sm \neq 0$ since $M \to M_{\mathfrak{p}}$ is monic and thus $s \notin \mathfrak{q}$. Therefore $\mathfrak{q} = \mathfrak{p}$.