MATH 612 **Due Wednesday April 14, 1993**

1. Let R and R' be rings.

a) Prove that there exists a multiplication on $R \otimes_{\mathbb{Z}} R'$ such that $(r_1 \otimes r_1')(r_2 \otimes r_2') = r_1 r_2 \otimes r_1' r_2'$. Thus $R \otimes_{\mathbb{Z}} R'$ can be considered as a ring.

b) Prove that if R and R' are commutative, then the diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \otimes_{\mathbb{Z}} R' \end{array}$$

is a push-out in the category of commutative rings. (Here the maps out of $\mathbb{Z}$ are the only ones possible and the bottom horizontal map, for instance, is given by $r \mapsto r \otimes 1$.)

2. Let A be an R-algebra and let $f \in R[X]$ and $B = R[X]/(f)$. By abuse of notation, we will also use f to denote its image in $A[X]$. Prove that $A \otimes_R B \approx A[X]/(f)$.

3. If $\mathbb{C}$ is the field of complex numbers, prove that $i \otimes 1 + 1 \otimes i$ is a zero divisor in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

4. Let E and F be extensions of a field k, let $m = [E:k]$ and $n = [F:k]$. (These may possibly be infinite cardinals.) Then we can think of $E \otimes_k F$ as either a k-algebra, an E-algebra, or an F-algebra.

a) Prove that $\dim_E E \otimes_k F = n$, $\dim_F E \otimes_k F = m$ and $\dim_k E \otimes_k F = mn$.

b) Prove that if $e_1, \ldots, e_r$ are elements of E linearly independant over k, and $f_1, \ldots, f_s$ are k-linearly independant elements of F, then the rs elements $e_1 \otimes f_1$, $e_1 \otimes f_2, \ldots, e_r \otimes f_s$ of $E \otimes_k F$ are linearly independant over k.

c) Prove that if $F \subseteq F'$ are fields then the inclusion $F \hookrightarrow F'$ induces a **monomorphism** $E \otimes_k F \to E \otimes_k F'$, so that it is reasonable to think of $E \otimes_k F$ as a k-subalgebra of $E \otimes_k F'$.

d) If E and F are both contained in an extension field K of k, prove that E and F are *linearly disjoint* over k [Lang, §X.5, p. 379] [Hungerford, §VI.2, p. 318] $\iff$ $E \otimes_k F$ is a field, and in this case $E \otimes_k F \approx EF$.