Lemma. Let L be a left R-module and M a right R-module . Let $\ell_1, \ldots, \ell_n \in L$ and $m_1, \ldots, m_n \in M$ be such that $\sum \ell_i \otimes m_i = 0 \in L \otimes_R M$. Then there exist **finitely generated** submodules $L_1 \subseteq L$ and $M_1 \subseteq M$ such that

$$\ell_1, \ldots, \ell_n \in L_1$$

$$m_1, \ldots, m_n \in M \quad \text{and}$$

$$\sum \ell_i \otimes m_i = 0 \in L_1 \otimes_R M_1.$$

(The fact that the tensor product is not left exact makes this lemma non-trivial.)

Definition. Let $\{M_i\}_{i \in I}$ be a family of submodules of an R-module M. We say that this family is **directed** if $(\forall i,\, j \in I)\,(\exists k \in I)\; M_i \cup M_j \subseteq M_k$.
In this case, $\bigcup_I M_i$ is called a **direct union**.

1. **a)** Prove that a direct union of submodules of M is a submodule.
b) Prove that a direct union of flat submodules of M is flat.
c) Prove that every R-module is a direct union of finitely generated submodules.
d) Prove that a torsion free module over a principal ideal domain is flat.

Definition. If $N \subseteq M$ are modules over a PID R then N is called a **pure** submodule of M if and only if for every $r \in R$, $N \cap rM = rN$.
(NOTE: This is **not** the correct definition over a general ring R.)

2. Let $N \subseteq M$ be modules over a PID R.
a) Prove that N is pure in M if and only if for every **cyclic** R-module L the induced map $L \otimes_R N \to L \otimes_R M$ is monic.
b) Prove that N is a pure submodule of M if and only if every coset $x \in M/N$ has a representative m' such that $\operatorname{ann} m' = \operatorname{ann} x$. (For abelian groups, we can translate this condition as "Every coset in M/N has a representative in M whose order is the same as the order of the coset.")
c) Prove that if M/N is torsion free then N is pure in M.

Definition. The **conjugacy class** of an element g in a group G is $\{h^{-1}gh \mid h \in G\}$. If G is infinite, then a conjugacy class may be either finite or infinite.

3. Prove that the center of a group ring $R[G]$ is free as an R-module with a basis consisting of those elements $\sum_C g$, where C ranges over the finite conjugacy classes of G.

For any group algebra $R[G]$ there exists a unique R-algebra morphism $\varepsilon\colon R[G] \to R$ such that $\varepsilon(g) = 1$ for all $g \in G$. This is called the **augmentation map**. $\operatorname{Ker}\varepsilon$ is called the **augmentation ideal** of $R[G]$.

4. **a)** Prove that the augmentation ideal is the ideal of $R[G]$ generated by all elements $g - 1$ for $g \in G$.

b) Let I be the augmentation ideal of $R[G]$. Prove that for $g_1, g_2 \in G$, $g_1 g_2 - 1 \equiv (g_1 - 1) + (g_2 - 1) \pmod{I^2}$.

c) Let $[G, G]$ be the **commutator subgroup** of G, i. e. the subgroup generated by all elements $g_1 g_2 g_1^{-1} g_2^{-1}$ and let I be the augmentation ideal of $\mathbb{Z}[G]$. Prove that $G/[G, G] \approx I/I^2$.

d) Prove that if G and G' are (not necessarily finite) **abelian** groups and $\mathbb{Z}[G] \approx \mathbb{Z}[G']$ then $G \approx G'$.

Lemma. Let G_1 and G_2 be groups and R a commutative ring. Let $\varepsilon_i\colon R[G_i] \to R$ be the augmentation ideals for the group rings and let $I_i = \operatorname{Ker} \varepsilon_i$. If $R[G_1] \approx R[G_2]$ as R-algebras then there exists an isomorphism $\theta\colon R[G_1] \to R[G_2]$ such that $\theta(I_1) = I_2$.

PROOF: Let $\zeta\colon R[G_1] \to R[G_2]$ be an isomorphism of R-algebras and let $\rho = \varepsilon_2\zeta$. Then ρ is a ring morphism from $R[G_1]$ to R, so for all $g \in G_1$, $\rho(g)$ is a unit in R. Now since $R[G_1]$ is a free R-module with basis G_1, there is a unique R-linear map $\theta\colon R[G_1] \to R[G_2]$ such that $\theta(g) = \rho(g)^{-1}\zeta(g)$. Notice that for $g, h \in G$, $\theta(gh) = \theta(g)\theta(h)$. It follows easily that θ is an isomorphism of R-algebras. Furthermore, for $g \in G_1$, $\varepsilon_2\theta(g) = \rho(g)^{-1}\varepsilon_1\zeta(g) = \rho(g)^{-1}\rho(g) = 1$. Thus $\theta(g-1) \subseteq \operatorname{Ker}\varepsilon_2 = I_2$ so $\theta(I_1) \subseteq I_2$. Thus θ induces an R-algebra morphism $\bar{\theta}\colon R[G_1]/I_1 \to R[G_2]/I_2$. Now if $x \in R$ then $x \equiv \varepsilon_1(x) \pmod{I_1}$ and since $\varepsilon_1(x) \in R$ it follows that $\bar{\theta}(x + I_1) = \theta(\varepsilon_1(x) + I_1) = \varepsilon_1(x) + I_2$. Thus $\bar{\theta}$ is an isomorphism, so $\theta(I_1) = I_2$. ☑

4. **a)** In fact, one sees that I is the **free** abelian group with basis $\{g-1 \mid g \in G\}$.

b) Clearly I^2 is generated by those elements of the form

$$(g_1 - 1)(g_2 - 1) = (g_1g_2 - 1) - (g_1 - 1) - (g_2 - 1).$$

c) Let $\varphi\colon G \to I/I^2$ be defined by $\varphi(g) = g - 1 + I^2$. By **b)** this is a homomorphism from G into the additive group I/I^2. Since the latter is abelian, it induces a homomorphism $\bar{\varphi}\colon G/[G,G]\colon \to I/I^2$.

On the other hand, since I is the free abelian group with basis consisting of those elements $g-1$ for $g \in G$ and since $G/[G,G]$ is an abelian group, there is a unique homomorphism $\psi\colon I \to G/[G,G]$ with $\psi(g-1) = g[G,G]$ (and taking addition to multiplication). Since clearly $\psi(\,(g_1g_2 - 1) - (g_1 - 1) - (g_2 - 1)\,) = 0$, ψ induces a homomorphism $\bar{\psi}\colon I/I^2 \to G/[G,G]$. It is readily apparent that $\bar{\varphi}$ and $\bar{\psi}$ are inverse to each other. ☑

Note. The trick above which changes multiplication into addition is worth remembering. The homomorphism $\varphi\colon g \mapsto g - 1$ could almost be thought of here as the algebraist's logarithm It has other applications beyond the context of group rings.