**1.** Let $k$ be a field. Suppose $p = \operatorname{char} k \neq 0$.

FACT: There exists an extension field of $k$ denoted $k^{\frac{1}{p}}$ which consists precisely of the $p^{\text{th}}$ roots of elements of $k$.

**2.** Prove that if $F$ is an algebraic extension of $k$ which is not separable, then $k^{\frac{1}{p}} \otimes_k F$ contains non-trivial nilpotent elements. In particular, $k^{\frac{1}{p}}$ is not linearly disjoint from $F$.

**a)** Prove that if $F$ is any extension of $k$ then the nil radical of $k^{\frac{1}{p}} \otimes_k F$ is the unique maximal ideal of $k^{\frac{1}{p}} \otimes_k F$: in fact, if $x = \sum e_i \otimes f_i \in k^{\frac{1}{p}} \otimes_k F$ and $x^p \neq 0$, then $x$ is invertible in $k^{\frac{1}{p}} \otimes_k F$.

**b)** Prove that if $F$ is a separable extension of $k$ and $E$ is any extension of $k$, then $E \otimes_k F$ has trivial nil radical. (HINT: Reduce to the case where $[F : k]$ is finite and also use Theorem 6.1, p. 290.)

**c)** Prove that if $F$ is a separable extension of $k$ then $k^{\frac{1}{p}} \otimes_k F$ is a field.