1. UNIVERSAL (AND COUNIVERSAL) CONSTRUCTIONS. [Hungerford, pp. 52–58].

a) Let K be a commutative ring. Recall the definition of a K-algebra [Hungerford, p. 227]. There is a functor from the category of sets to the category of commutative K-algebras that takes a set S to the ring of polynomials in the set of variables $\{X_s\}_{s\in S}$. We denote this polynomial ring by $K[X_s]_{s\in S}$. (Recall that even though S may be infinite, any specific polynomial in $K[X_s]_{s\in S}$ has only finitely many non-trivial coefficients.)

Show that if A is any commutative K-algebra and $f\colon S \to A$ any *function*, then there is a unique morphism of K-algebras $\tilde{f}\colon K[X_s] \to A$ such that $\tilde{f}(X_s) = f(s)$ for all $s \in S$. Show further that this universal property characterizes the functor $S \mapsto K[X_s]_{s\in S}$.

(This means that $K[X_s]_{s\in S}$ is the **free K-algebra on the set S** (c.f. Hungerford, p. 55).)

b) Explain why if I is an ideal in a ring R then the category of R/I-modules can be essentially identified with a subcategory of the category of all R-modules, namely those R-modules M such that $IM = 0$.

Now suppose (for convenience) that R is commutative and let $\mathfrak{a}$ be an ideal. Let $\mathcal{C}$ be the category of $R/\mathfrak{a}$-modules. There are two functors that take the category of R-modules into $\mathcal{C}$. One takes an R-module M to $M[\mathfrak{a}]$ (as defined in class) and the other takes M to $M/\mathfrak{a}M$.

Characterize each of these functors by a universal (or couniversal) construction.

2. **a)** If N is a submodule of an R-module M and S is a multiplicative set in the center of R, then $S^{-1}N$ can be identified as a submodule of $S^{-1}M$, and $S^{-1}M/S^{-1}N \approx S^{-1}(M/N)$.

b) If I is a (two-sided) ideal in R and $\bar{S} = \{s + I \mid s \in S\} \subseteq R/I$, then $S^{-1}I$ is an ideal in $S^{-1}R$ and the rings $S^{-1}R/S^{-1}I$ and $\bar{S}^{-1}(R/I)$ are isomorphic.

3. Prove that if R is noetherian then $S^{-1}R$ is noetherian. (NOTE: R need not be commutative but, as always, $S \subseteq \text{Center}\, R$.)

4. Let M be a module over a commutative ring R, let $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ a prime ideal. Prove that if $\mathfrak{a} \not\subseteq \mathfrak{p}$ then $M_{\mathfrak{p}} = \mathfrak{a}M_{\mathfrak{p}}$

5. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in a commutative ring R such that $\mathfrak{a}_i + \mathfrak{a}_j = R$ for all $i \neq j$, and let M be an R-module. For all $1 \le k \le n$, $\cap_I\, \mathfrak{a}_i M \subseteq \mathfrak{a}_k M$, so there is a quotient map $\gamma_k\colon \dfrac{M}{\bigcap \mathfrak{a}_i M} \to \dfrac{M}{\mathfrak{a}_k M}$. Use localization (and problem 4) to prove the

Chinese Remainder Theorem: The map $\gamma\colon \dfrac{M}{\bigcap \mathfrak{a}_i M} \to \bigoplus_I \dfrac{M}{\mathfrak{a}_i M}$ induced by the family of maps γ_k is an isomorphism.
(Restated: For every family $m_1, \ldots, m_n \in M$ there exists $m \in M$ such that $m \equiv m_i \pmod{\mathfrak{a}_i M}$ for all i. Furthermore, m is uniquely determined modulo $\bigcap \mathfrak{a}_i M$.)

Recall that if M is a module over a commutative ring R and $\mathfrak{p}$ is a **prime** ideal of R then we write $\mathfrak{p} \in \operatorname{Ass} M$ if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{ann} m$.
(This is equivalent to the condition that M contains a submodule isomorphic to $R/\mathfrak{p}$.)
Furthermore, we say that M is **$\mathfrak{p}$-primary** if $\operatorname{Ass} M = \{\mathfrak{p}\}$.

6. Prove that if $\mathfrak{p}$ if a prime ideal in a commutative ring R then $R/\mathfrak{p}$ is $\mathfrak{p}$-primary.

7. Let R be a commutative ring and N a submodule of an R-module M.
Prove that $\operatorname{Ass} N \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass}(M/N)$.

Lemma. Suppose that M is a module over a commutative noetherian ring R and let $m \neq 0 \in M$. Then there exists $\mathfrak{p} \in \operatorname{Ass} M$ such that $\operatorname{ann} m \subseteq \mathfrak{p}$. In fact, if $\mathfrak{q}$ is any prime ideal such that $\operatorname{ann} m \subseteq \mathfrak{q}$ then there exists $\mathfrak{p} \in \operatorname{Ass} M$ with $\mathfrak{p} \subseteq \mathfrak{q}$.

PROOF: Since $\operatorname{ann} m \neq R$ then by Zorn's Lemma there exist prime ideals $\mathfrak{q}$ (in fact maximal ideals $\mathfrak{q}$) with $\operatorname{ann} m \subseteq \mathfrak{q}$. Let $\mathfrak{q}$ be such a prime ideal. Since R is noetherian, among the ideals $\mathfrak{p}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} = \operatorname{ann} rm$ for $r \in R$ and $rm \neq 0$, there exist ones maximal with this property. Let $\mathfrak{p} = \operatorname{ann} rm$ be such an ideal. We claim that $\mathfrak{p}$ is prime. In fact, if $r_1 r_2 \in \mathfrak{p}$ and $r_2 \notin \mathfrak{p}$ then $r_2 rm \neq 0$ and $\mathfrak{p} \subseteq \mathfrak{p} + (r_1) \subseteq \operatorname{ann}(r_2 rm)$. By the maximality of $\mathfrak{p}$ we conclude that $\mathfrak{p} + (r_1) = \mathfrak{p}$, i.e. $r_1 \in \mathfrak{p}$. This shows that $\mathfrak{p}$ is prime and since $\mathfrak{p} = \operatorname{ann} rm$, thus $\mathfrak{p} \in \operatorname{Ass} M$. ☑

8. Suppose that R is a commutative noetherian ring. Prove that an R-module M is $\mathfrak{p}$-primary if and only if the natural map $\theta\colon M \to M_{\mathfrak{p}}$ is monic and

$$(\forall m \in M)\,(\exists k \geq 1)\ \ \mathfrak{p}^k m = 0.$$

February 3 Answers

1. b)

$$\begin{array}{ccc} M & & \\ \big\downarrow & & \\ M/\mathfrak{a}M & \longrightarrow & N \end{array} \qquad\qquad \begin{array}{ccc} P & \longrightarrow & M[\mathfrak{a}] \\ & & \big\downarrow \\ & & M \end{array}$$

Here, N and P are R-modules such that $\mathfrak{a}N = \mathfrak{a}P = 0$. Whenever there is a slanted map as indicated, there is a unique horizontal map making the diagram commute.

5. Chinese Remainder Theorem. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in a commutative ring R such that $\mathfrak{a}_i + \mathfrak{a}_j = R$ for all $i \neq j$. Let M be an R-module. Then

$$\frac{M}{\bigcap \mathfrak{a}_i M} \approx \bigoplus_1^n \frac{M}{\mathfrak{a}_i M}.$$

PROOF: For each k, $\bigcap \mathfrak{a}_i M \subseteq \mathfrak{a}_k M$. Thus there is an obvious map $M/\bigcap \mathfrak{a}_i M \to M/\mathfrak{a}_k M$ and therefore there is a map $M/\bigcap \mathfrak{a}_i M \to \bigoplus M/\mathfrak{a}_k M$. Since $\mathfrak{a}_i + \mathfrak{a}_j = R$ for $k \neq j$, a prime ideal $\mathfrak{p}$ can contain at most one $\mathfrak{a}_i$, Thus if $\mathfrak{a}_k \subseteq \mathfrak{p}$ then $(\bigcap_1^n \mathfrak{a}_i M)_{\mathfrak{p}} = \mathfrak{a}_k M_{\mathfrak{p}}$ and $(\bigoplus_1^n \mathfrak{a}_i M)_{\mathfrak{p}} = \mathfrak{a}_k M_{\mathfrak{p}}$. Therefore for all $\mathfrak{p}$,

$$\left(\frac{M}{\mathfrak{a}_k M}\right)_{\mathfrak{p}} = \left(\frac{M}{\bigcap_1^n \mathfrak{a}_i M}\right)_{\mathfrak{p}} \to \left(\bigoplus_1^n \frac{M}{\mathfrak{a}_i M}\right)_{\mathfrak{p}} = \left(\frac{M}{\mathfrak{a}_k M}\right)_{\mathfrak{p}}$$

is an isomorphism, so the result follows from a result proved in class. ☑

7. If $N \subseteq M$ then $\operatorname{Ass} N \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass} M/N$.

PROOF: If $\mathfrak{p} \in \operatorname{Ass} N$ then $\mathfrak{p} = \operatorname{ann} n$ for $n \in N$ and clearly $\mathfrak{p} \in \operatorname{Ass} M$. Now suppose that $\mathfrak{p} \in \operatorname{Ass} M$. Then M contains a submodule M_1 isomorphic to $R/\mathfrak{p}$. By Problem **6**, $\operatorname{Ass} M_1 = \{\mathfrak{p}\}$ and so if $M_1 \cap N \neq 0$ then $\varnothing \neq \operatorname{Ass} M_1 \cap N \subseteq \{\mathfrak{p}\}$, so $\mathfrak{p} \in \operatorname{Ass} M_1 \cap N \subseteq \operatorname{Ass} N$. On the other hand, if $M_1 \cap N = 0$ then M/N contains the submodule $(M_1 + N)/N \approx M_1 \approx R/\mathfrak{p}$, so $\mathfrak{p} \in \operatorname{Ass} M/N$. ☑

8. If M is an R-module and $\mathfrak{p}$ is a prime ideal, the following conditions are equivalent:

(1) M is $\mathfrak{p}$-primary.
(2) The natural map $\theta\colon M \to M_{\mathfrak{p}}$ is monic and

$$(\forall r \in \mathfrak{p})\,(\forall m \in M)\,(\exists k \geq 1)\;\; r^k m = 0.$$

PROOF: ($\Rightarrow$): Suppose that $\operatorname{Ass} M = \{\mathfrak{p}\}$. Then for every $m \neq 0 \in M$, by the lemma preceding the problem, $\operatorname{ann} m \subseteq \mathfrak{p}$ and every prime ideal containing $\operatorname{ann} m$ contains $\mathfrak{p}$. From this it follows first that if $s \notin \mathfrak{p}$ then $sm \neq 0$, showing that $m \notin \operatorname{Ker}\theta$, where $\theta\colon M \to M_{\mathfrak{p}}$ is the canonical map. Furthermore, let $r \neq 0 \in \mathfrak{p}$ and let $S = \{r^k \mid k \geq 1\}$. If $S \cap \operatorname{ann} m = \varnothing$ then by a result proved in class there exists a prime ideal $\mathfrak{q}$ with $\mathfrak{q} \supseteq \operatorname{ann} m$ and $\mathfrak{q} \cap S = \varnothing$. But then $\mathfrak{q} \supseteq \mathfrak{p}$ and $r \notin \mathfrak{q}$, a CONTRADICTION. Thus there exists an element $r^k \in S \cap \operatorname{ann} m$, so $r^k m = 0$. Since $\mathfrak{p}$ is finitely generated (because R is noetherian) it then follows easily that $\mathfrak{p}^{k'} m = 0$ for some k'.

($\Leftarrow$): Now suppose the stated conditions hold and let $\mathfrak{q} \in \operatorname{Ass} M$. Then $\mathfrak{q} = \operatorname{ann} m$ for some m. By assumption, for some k, $\mathfrak{p}^k \subseteq \operatorname{ann} m = \mathfrak{q}$ Then $\mathfrak{p} \subseteq \mathfrak{q}$ since $\mathfrak{q}$ is prime. On the other hand, if $s \notin \mathfrak{p}$ then $sm \neq 0$ since $M \to M_{\mathfrak{p}}$ is monic and thus $s \notin \mathfrak{q}$. Therefore $\mathfrak{q} = \mathfrak{p}$. ☑