1. TWO MORE UNIVERSAL (AND COUNIVERSAL) CONSTRUCTIONS.

Consider a square
$$\begin{array}{ccc} L & \xrightarrow{\varphi'} & N \\ \big\downarrow{\scriptstyle \psi'} & & {\scriptstyle \psi}\big\downarrow \\ M & \xrightarrow{\varphi} & P \end{array} .$$

Define $\sigma\colon L \to M \oplus N$ and $\tau\colon M \oplus N \to P$ by
$$\sigma(\ell) = (\psi'(\ell),\, -\varphi'(\ell)) \quad \text{and} \quad \tau(m,n) = \varphi(m) + \psi(n).$$

a) Prove that the square commutes if and only if $\tau\sigma = 0$.

b) Prove that the following conditions are equivalent:

(1) $0 \to L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P$ is exact.

(2) The square commutates and
$(\forall m \in M,\, n \in N) \quad [\ \varphi(m) = \psi(n) \iff (\exists!\, \ell \in L)\ n = \varphi'(\ell),\ m = \psi'(\ell)\ ]$.

(3) The square commutes and whenever $\alpha\colon X \to M$ and $\beta\colon X \to N$ are maps (for any R-module X) such that $\varphi\alpha = \psi\beta$, then there exists a unique map $\theta\colon X \to L$ such that $\alpha = \psi'\theta$ and $\beta = \varphi'\theta$.

$$\begin{array}{ccccccc} X & & & & L & \xrightarrow{\varphi'} & N \\ & & & & {\scriptstyle \psi'}\big\downarrow & & {\scriptstyle \psi}\big\downarrow \\ & L & \xrightarrow{\varphi'} & N & M & \xrightarrow{\varphi} & P \\ & {\scriptstyle \psi'}\big\downarrow & & {\scriptstyle \psi}\big\downarrow & & & \\ & M & \xrightarrow{\varphi} & P & & & Y \end{array}$$

c) Prove that the following conditions are equivalent:

(1) $L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P \to 0$ is exact.

(2) The square commutes, $P = \varphi(M) + \psi(N)$ and
$(\forall m \in M,\, n \in N) \quad [\ \varphi(m) = \psi(n) \iff (\exists \ell \in L)\ n = \varphi'(\ell),\ m = \psi'(\ell)\ ]$.
(Note that in this case ℓ need not be unique.)

(3) The square commutes and whenever $\gamma\colon M \to Y$ and $\delta\colon N \to Y$ are maps (for any R-module Y) such that $\gamma\psi' = \delta\varphi'$, then there exists a unique map $\zeta\colon P \to Y$ such that $\gamma = \zeta\varphi$ and $\delta = \zeta\psi$.

Definition. If the equivalent conditions in **b)** are satisfied, we say that the square above is a **pull-back** (Hungerford, p. 484).
If the conditions in **c)** are satisfied, we say that it is a **push-out**.

2. **a)** Prove that if the square in problem **1** is a pull-back, then $\operatorname{Ker}\varphi' \approx \operatorname{Ker}\varphi$.

b) Prove that if the square in problem **1** is a push-out, then $\operatorname{Coker}\varphi' \approx \operatorname{Coker}\varphi$. (NOTE: $\operatorname{Coker}\varphi = P/\varphi(M)$.)

c) Show that Noether's Second Isomorphism Theorem (Hungerford, Theorem 1.9 (i), p. 173) is a special case of part **b)**.

3. Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{\varphi'} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{\varphi} & P & \longrightarrow & 0 \end{array}$$

Prove that the right hand square is both a pull-back and a push-out.

4. Let S be a multiplicative set in a commutative noetherian ring R and let M be an R-module. Prove that

$$\operatorname{Supp}_{S^{-1}R} S^{-1}M = \{\, \mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \operatorname{Supp} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\}$$
$$\operatorname{Ass}_{S^{-1}R} S^{-1}M = \{\, \mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \operatorname{Ass} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\}$$
$$\operatorname{Ass}_R S^{-1}M = \{\, \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M \quad \& \quad \mathfrak{p} \cap S = \varnothing \,\}.$$

5. Let M be a module over a commutative noetherian ring R such that $\operatorname{Ass} M$ consists of maximal ideals.

a) Prove that $\operatorname{Ass} M = \operatorname{Supp} M$.

b) Prove for every $\mathfrak{p} \in \operatorname{Ass} M$, the canonical map $M \to M_{\mathfrak{p}}$ is a surjection and

$$M_{\mathfrak{p}} \approx \{m \in M \mid (\exists k)\ \mathfrak{p}^k m = 0\}.$$

c) Prove that the family of maps $M \to M_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Ass} M$ induces an isomorphism

$$M \xrightarrow{\approx} \bigoplus_{\operatorname{Ass} M} M_{\mathfrak{p}}.$$

1.

Consider a square $\begin{array}{ccc} L & \xrightarrow{\varphi'} & N \\ \downarrow{\scriptstyle \psi'} & & {\scriptstyle \psi}\downarrow \\ M & \xrightarrow{\varphi} & P \end{array}$.

Define $\sigma\colon L \longrightarrow M \oplus N$ and $\tau\colon M \oplus N \longrightarrow P$ by

$$\sigma(\ell) = (\psi'(\ell),\, -\varphi'(\ell)) \quad \text{and} \quad \tau(m,n) = \varphi(m) + \psi(n).$$

c) (1) $L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P \to 0$ is exact.

(3) The square commutes and whenever $\gamma\colon M \to Y$ and $\delta\colon N \to Y$ are maps (for any R-module Y) such that $\gamma\psi' = \delta\varphi'$, then there exists a unique map $\zeta\colon P \to Y$ such that $\gamma = \zeta\varphi$ and $\delta = \zeta\psi$.

PROOF: (3) $\Rightarrow$ (1): *Proof that* $\operatorname{Ker}\tau \subseteq \sigma(L)$*:*

Note that $\sigma(L) = \{(\psi'(\ell), -\varphi'(\ell)) \mid \ell \in L\}$.

Consider the following square:

$$\begin{array}{ccc} L & \xrightarrow{\varphi'} & N \\ {\scriptstyle \psi'}\downarrow & & {\scriptstyle \psi}\downarrow \\ M & \xrightarrow{\varphi} & P \end{array}$$

$$\frac{M \oplus N}{\sigma(L)}\,,$$

where $\gamma(m) = (m,0) + \sigma(L)$ and $\delta(n) = (0,n) + \sigma(L)$. Note that

$$\gamma\psi'(\ell) - \delta\varphi'(\ell) = (\psi'(\ell),\, -\varphi'(\ell)) + \sigma(L) = 0 \in (M \oplus N)/\sigma(L)\,,$$

so by hypothesis there exists ζ making the diagram commute. Now suppose that $\tau(m,n) = \varphi(m) + \psi(n) = 0$. Then

$$(m,n) + \sigma(L) = \gamma(m) + \delta(n) = \zeta\varphi(m) + \zeta\psi(n) = \zeta(\varphi(m) + \psi(n)) = 0$$

so $(m,n) \in \sigma(L)$. $\boxed{\checkmark}$

2. b)

$$\begin{array}{ccccccc}
L & \xrightarrow{\varphi'} & N & \xrightarrow{\gamma'} & C & \longrightarrow & 0 \\
{\scriptstyle\psi'}\downarrow & & {\scriptstyle\psi}\downarrow & & {\scriptstyle\mu}\downarrow & & \\
M & \xrightarrow{\varphi} & P & \xrightarrow{\xi} & D & \longrightarrow & 0
\end{array}$$

Let $C = \operatorname{Coker}\varphi'$ and $D = \operatorname{Coker}\varphi$. Now $(\xi\psi)\varphi' = \xi\varphi\psi' = 0$ so by the Induced Homomorphism Theorem there exists a unique map $\mu\colon C \to D$ making the above diagram commute.
On the other hand, since $\gamma'\varphi' = 0 = 0\psi'$, by the categorical definition of a push-out there exists a unique map $\zeta\colon P \to C$ with $\zeta\psi = \gamma'$ and $\zeta\varphi = 0$. Again by the Induced Homomorphism Theorem ζ induces a map $\eta\colon D \to C$ such that $\eta\xi\psi = \zeta\psi = \gamma'$. Then $(\mu\eta\xi)\psi = \mu\zeta\psi = \mu\gamma' = \xi\psi$ and $(\mu\eta\xi)\varphi = 0 = \xi\varphi$, so by the definition of a push-out it follows that $\mu\eta\xi = \xi$, and thus $\mu\eta = 1_D$ because ξ is an epimorphism. Also $(\eta\mu)\gamma' = \eta\xi\psi = \zeta\psi = \gamma'$ so $\eta\mu = 1_C$ because γ' is an epimorphism. Thus $C \approx D$. ☑

3.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\eta'} & L & \xrightarrow{\varphi'} & N & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle\psi'}\downarrow & & {\scriptstyle\psi}\downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{\eta} & M & \xrightarrow{\varphi} & P & \longrightarrow & 0
\end{array}$$

Proof that the square is a push-out: Let $\gamma\colon M \to Y$ and $\delta\colon N \to Y$ be such that $\gamma\psi' = \delta\varphi'$. Then $\gamma\eta = \gamma\psi'\eta' = \delta\varphi'\eta' = 0$, so by the Induced Homomorphism Theorem there exists a unique $\zeta\colon P \to Y$ with $\zeta\varphi = \gamma$. Furthermore $(\zeta\psi)\varphi' = \zeta\varphi\psi' = \gamma\psi' = \delta\varphi'$. Since φ' is an epimorphism, we conclude that $\zeta\psi = \delta$. Thus the square in question satisfies the categorical definition of a push-out.
Proof that the square is a pull-back: (Actually, knowing that the square is a push-out, we are already half-way to proving it is a pull-back. But we will start from scratch.) Suppose $m \in M$ and $n \in N$ with $\varphi(m) = \psi(n)$. Since φ' is epic,

$$(\exists \ell \in L) \quad n = \varphi'(\ell).$$

Then $\varphi(m - \psi'(\ell)) = \varphi(m) - \varphi\psi'(\ell) = \psi(n) - \psi\varphi'(\ell) = 0$. Therefore $m - \psi'(\ell) \in \operatorname{Ker}\varphi$ so by exactness there exists a unique $k \in K$ with $m - \psi'(\ell) = \eta(k) = \psi'\eta'(k)$. Thus

$$m = \psi'(\eta'(k) + \ell) \quad \text{and} \quad n = \varphi'(\eta'(k) + \ell).$$

Furthermore, $\eta'(k) + \ell$ is the unique element in L that works. In fact if $m = \psi'(\ell')$ and $n = \varphi'(\ell')$, then $\varphi'(\eta'(k) + \ell - \ell') = 0$ so by exactness

$$(\exists k' \in K) \quad \eta'(k) + \ell - \ell' = \eta'(k')$$

and so $0 = \psi'(\eta'(k) + \ell - \ell') = \psi'\eta'(k') = \eta(k')$ and so $k' = 0$ because η is monic, so $\ell' = \eta'(k) + \ell$.

4. There are three relevant observations:
(1) The primes of $S^{-1}R$ are precisely the ideals of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \cap S = \varnothing$. (If $\mathfrak{p} \cap S \neq \varnothing$ then $S^{-1}\mathfrak{p} = R$.)
(2) If $\mathfrak{p}$ is a prime in R then $S^{-1}\mathfrak{p} \subseteq S^{-1}R$ and in fact $S^{-1}\mathfrak{p} = \mathfrak{p}S^{-1}R$.
(3) If $S \cap \mathfrak{p} = \varnothing$ then $S \subseteq R \setminus \mathfrak{p}$ and so $S^{-1}M_{\mathfrak{p}} \approx M_{\mathfrak{p}} \approx S^{-1}M_{S^{-1}\mathfrak{p}}$ (where the LHS can be interpreted in two ways and these two interpretations agree).
We now see immediately that if $\mathfrak{p} \cap S = \varnothing$ then $S^{-1}M_{S^{-1}\mathfrak{p}} \neq 0$ and if only if $M_{\mathfrak{p}} \neq 0$.

Another observation: If $m \in M$, then the annihilator in $S^{-1}R$ of $m/1$ is $S^{-1}(\operatorname{ann} m) \subseteq S^{-1}R$. In fact, if $r \in \operatorname{ann} m$ then $rm = 0$ and for any $s \in S$, $(r/s)(m/1) = rm/s = 0$, so $r/s \in \operatorname{ann}_{S^{-1}R}(m/1)$. Conversely, if $(r/s)(m/1) = 0$ then $s'rm = 0$ for some $s' \in S$. Then $s'r \in \operatorname{ann} m$ and $r/s = s'r/s's \in S^{-1}(\operatorname{ann} m)$.
Now let $\mathfrak{p} \in \operatorname{Ass} M$ and $\mathfrak{p} \cap S = \varnothing$. Then $\mathfrak{p} = \operatorname{ann} m$ for some $m \in M$ and so $S^{-1}\mathfrak{p} = \operatorname{ann}(m/1) \in \operatorname{Ass}_{S^{-1}R} S^{-1}M$.
Conversely, a prime $\mathfrak{P} = \operatorname{ann}_{S^{-1}R}(m/s)$ in $\operatorname{Ass}_{S^{-1}R} S^{-1}M$ must have the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} = \theta_R^{-1}(\mathfrak{P})$. Thus for each $p \in \mathfrak{p}$ there exists $s' \in S$ with $s'pm = 0$. Since $\mathfrak{p}$ is finitely generated, there exists $s'' \in S$ with $s''\mathfrak{p}\, m = \mathfrak{p}s''m = 0$. Thus $\mathfrak{p} \subseteq \operatorname{ann} s''m$. On the other hand, if $r \in \operatorname{ann} s''m$ then $rs''m = 0$ and so $\dfrac{rs''}{s''}\dfrac{m}{s} = 0$ so $r/1 \in S^{-1}\mathfrak{p}$ and thus $r \in \theta_R^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$. Therefore $\mathfrak{p} = \operatorname{ann} s''m$.

Now suppose $\mathfrak{p} \cap S = \varnothing$ and $\mathfrak{p} \in \operatorname{Ass} R$ and $\mathfrak{p} = \operatorname{ann} m$. We claim that $\mathfrak{p} = \operatorname{ann}(m/1)$. In fact, for $r \in R$, $r \in \operatorname{ann}(m/1)$ if and only if $srm = 0$ for some $s \in S$, and this holds if and only if $sr \in \operatorname{ann} m = \mathfrak{p}$ for some $s \in S$, but this is true if and only if $r \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $s \notin \mathfrak{p}$. Thus $\mathfrak{p} \in \operatorname{Ass}_R S^{-1}M$.
Conversely, suppose $\mathfrak{p}$ is a prime in R and $\mathfrak{p} \in \operatorname{Ass} S^{-1}M$. Then $\mathfrak{p} = \operatorname{ann}(m/1)$ for $m/1 \in S^{-1}M$. Since $m/1 \neq 0$ (otherwise $\operatorname{ann}(m/1) = R$), it follows that $sm \neq 0$ for all $s \in S$ and thus $\mathfrak{p} \cap S = \varnothing$. Furthermore for each $p \in \mathfrak{p}$, $pm/1 = 0$ so $spm = 0$ for some $s \in S$. Since $\mathfrak{p}$ is finitely generated it follows that there exists $s \in S$ such that $s\mathfrak{p}m = \mathfrak{p}sm = 0$. Therefore $\mathfrak{p} \subseteq \operatorname{ann}(sm)$. On the other hand, clearly $\operatorname{ann}(sm) \subseteq \operatorname{ann}\left(\dfrac{sm}{s}\right) = \mathfrak{p}$. So $\mathfrak{p} = \operatorname{ann} sm \in \operatorname{Ass} M$. ☑

5. **b)** To see that $M \to M_{\mathfrak{p}}$ is a surjection it suffices to see that for all prime ideals $\mathfrak{q}$, the induced map $M_{\mathfrak{q}} \to (M_{\mathfrak{p}})_{\mathfrak{q}}$ is a surjection. This is clear if $\mathfrak{p} \not\subseteq \mathfrak{q}$ since in that case by problem **4**, $\mathfrak{p} \notin \operatorname{Ass} M_{\mathfrak{q}} = \operatorname{Supp} M_{\mathfrak{q}}$ and so $(M_{\mathfrak{p}})_{\mathfrak{q}} = (M_{\mathfrak{q}})_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is maximal, this leaves only the case $\mathfrak{q} = \mathfrak{p}$, which is trivial.
A More Straightforward Proof: Let $m/s \in M_{\mathfrak{p}}$, where $s \notin \mathfrak{p}$. By Problem **3**, $\operatorname{Ass} M_{\mathfrak{p}} = \{\mathfrak{p}\}$, i.e. $M_{\mathfrak{p}}$ is $\mathfrak{p}$-primary, so by a previous homework problem $\mathfrak{p}^k(m/1) = 0$

for some $k \geq 1$. Now since $\mathfrak{p}$ is maximal, the only prime containing $\mathfrak{p}^k$ is $\mathfrak{p}$. It follows that $R/\mathfrak{p}^k$ is a local ring with maximal ideal $\mathfrak{p}/\mathfrak{p}^k$. Since $s \notin \mathfrak{p}$, thus the image of s in $R/\mathfrak{p}^k$ is invertible. Hence there exists $s' \notin \mathfrak{p}$ such that $ss' \equiv 1 \pmod{\mathfrak{p}^k}$. Then $(ss'-1)\,m = 0$ so that in $M_{\mathfrak{p}}$,

$$\frac{m}{s} = \frac{s'm}{1} = \theta(s'm),$$

showing that $\theta\colon M \to M_{\mathfrak{p}}$ is surjective.

Now let $M(\mathfrak{p}) = \{m \in M \mid (\exists k)\,\mathfrak{p}^k m = 0\}$. As seen in the previous paragraph, if $s \notin \mathfrak{p}$ then s is invertible modulo $\mathfrak{p}^k$ for all k, so if $m \in M(\mathfrak{p})$ then $sm \neq 0$ for all $s \notin \mathfrak{p}$ and therefore $m/1 \neq 0 \in M_{\mathfrak{p}}$. This shows that $\theta\colon M \to M_{\mathfrak{p}}$ restricts to a monomorphism from $M(\mathfrak{p})$ into $M_{\mathfrak{p}}$.

Now let $m/1 \in M_{\mathfrak{p}}$. As previously noted, there exists $k \geq 1$ such that $\mathfrak{p}^k(m/1) = 0$. Thus for each $r \in \mathfrak{p}^k$, $rm/1 = 0$ so there exists $s \notin S$ such that $srm = 0$. Since $\mathfrak{p}$ is finitely generated, it follows that there exists $s \notin \mathfrak{p}$ with $s\mathfrak{p}^k m = \mathfrak{p}^k sm = 0$, showing that $sm \in M(\mathfrak{p})$. . Furthermore, as previously seen, there exists $s' \notin \mathfrak{p}$ such that $ss' \equiv 1 \pmod{\mathfrak{p}^k}$. Then $ss'm \in M(\mathfrak{p})$ and

$$\frac{m}{1} = \frac{ss'm}{1}.$$

Therefore $\theta(M(\mathfrak{p})) = M_{\mathfrak{p}}$ and so $M_{\mathfrak{p}} \approx M(\mathfrak{p})$.

c) The family of maps $M \to M_{\mathfrak{p}}$ induces

$$\zeta\colon M \to \prod_{\operatorname{Ass} M} M_{\mathfrak{p}}$$

(where each coordinate of $\zeta(m)$ is given by $m/1 \in M_{\mathfrak{p}}$). Now note that for any $m \in M$, $\operatorname{Supp} Rm = \operatorname{Ass} Rm$ is finite, i.e. there are only finitely many prime ideals $\mathfrak{p}$ such that $m/1 \neq 0 \in M_{\mathfrak{p}}$. This shows that the image of ζ is in fact contained in $\bigoplus_{\operatorname{Ass} M} M_{\mathfrak{p}}$. It now suffices to see that for every prime ideal $\mathfrak{q}$, the induced map

$$M_{\mathfrak{q}} \to \bigoplus_{\operatorname{Ass} M} (M_{\mathfrak{p}})_{\mathfrak{q}}$$

is an isomorphism. But as seen in part **b)**, this reduces to the identity map $M_{\mathfrak{q}} \to M_{\mathfrak{q}}$.

A more conventional proof: It is easy to see that the family of submodules $M(\mathfrak{p})$ of M is independent, so $\bigoplus_{\operatorname{Ass} M} M(\mathfrak{p}) \subseteq M$. Now let $m \in M$. For each of the finitely many primes $\mathfrak{p}_i$ such that $m/1 \neq 0 \in M_{\mathfrak{p}_i}$ then exists k_i such that $\mathfrak{p}_i^{k_i} m/1 = 0 \in M_{\mathfrak{p}_i}$. Then $\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_n^{k_n} m = 0$. Now since $\mathfrak{p}_i + \mathfrak{p}_j = R$ for $i \neq j$, no maximal ideal can contain all the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, where

$$\mathfrak{a}_i = \mathfrak{p}_1^{k_1} \cdots \widehat{\mathfrak{p}_i^{k_i}} \cdots \mathfrak{p}_n^{k_n},$$

so $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n = R$ and there exist elements $a_i \in \mathfrak{a}_i$ with $a_1 + \cdots + a_n = 1$. Furthermore, $\mathfrak{p}^{k_i}\mathfrak{a}_i m = 0$ so $a_i m \in \mathfrak{a}_i m \subseteq M(\mathfrak{p}_i)$. Thus $m = a_1 m + \cdots + a_n M \in \bigoplus_{\operatorname{Ass} M} M(\mathfrak{p})$, showing that $M = \bigoplus M(\mathfrak{p})$. ☑