Definition. A family $\mathcal{F}$ of subsets of a set X is called a **filter** on X if it satisfies the following three conditions:

(1) $\varnothing \notin \mathcal{F}$.

(2) $F_1, F_2 \in \mathcal{F} \;\Rightarrow\; F_1 \cap F_2 \in \mathcal{F}$.

(3) $F_1 \in \mathcal{F}$ and $F_1 \subseteq F \subseteq X \;\Rightarrow\; F \in \mathcal{F}$.

A filter $\mathcal{F}$ which is maximal with these properties is called an **ultrafilter**. An ultrafiler $\mathcal{F}$ is called **principal** if $\bigcap \mathcal{F} \neq \varnothing$. (In this case, $\bigcap \mathcal{F} = \{x\}$ for some $x \in X$ and $\mathcal{F} = \{F \subseteq X \mid x \in F\}$.)

Lemma. A filter $\mathcal{F}$ is an ultrafilter if and only if for all $F \subseteq X$, either $F \in \mathcal{F}$ or $X \smallsetminus F \in \mathcal{F}$.

1. Let K be a field and X an infinite set.

a) Complete the proof given in class showing that there is a one-to-one correspondence between the ideals in $\prod_X K$ and the filters on X, and that prime ideals correspond to ultrafilters.

b) Characterize those primes corresponding to principal ultrafilters.

c) Show that all prime ideals in $\prod_X K$ are maximal.

2. Prove that the following conditions are equivalent for a commutative ring R:

(1) R has no non-trivial nilpotent elements and every prime ideal in R is maximal.

(2) For every prime ideal $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ is a field.

(3) For every $r \in R$ there exists $x \in R$ with $r^2 x = r$.

(4) Every finitely generated ideal in R is generated by an idempotent.

(HINT: This problem uses several different pieces of the theory we've developed.)

3. Prove that if $\mathfrak{p}$ is an ideal with height 0 in a commutative (not necessarily noetherian) ring R, then $\mathfrak{p}$ consists of zero divisors.

4. Suppose that $\mathfrak{p}_0$ and $\mathfrak{p}$ are prime ideals in $\mathbb{Z}[X]$ such that $0 \subsetneq \mathfrak{p}_0 \subsetneq \mathfrak{p}$.

a) Identify $\mathbb{Z}$ as a subring of $\mathbb{Z}[X]$ in the obvious way. Prove that $\mathfrak{p} \cap \mathbb{Z} \neq 0$.

b) Prove that there exists a prime number $p \in \mathbb{Z}$ such that $p \in \mathfrak{p}$.

c) Prove that $\mathfrak{p}$ is a maximal ideal.