MATH 612 **Due Wednesday March 3, 1993**

Some Principles of Diagram Chasing

Definition. Let $\varphi\colon A \to B$ be a morphism in a category $\mathcal{C}$.

(a) One says that φ is a **monomorphism** in $\mathcal{C}$ if whenever $\alpha, \beta : X \to A$ and $\varphi\alpha = \varphi\beta$ then $\alpha = \beta$.

(b) One says that φ is an **epimorphism** in $\mathcal{C}$ if whenever $\gamma, \delta\colon B \to Y$ and $\gamma\varphi = \delta\varphi$ then $\gamma = \delta$.

$$\begin{array}{ccc} \text{X} & & \text{Y} \\ & A \xrightarrow{\varphi} B & \\ \text{X} & & \text{Y} \end{array}$$

1. **a)** In the category of R-modules, show that φ is a monomorphism if and only if it is one-to-one and φ is an epimorphism if and only if it is surjective.

b) Let S be a multiplicative set in the center of a ring R. In the category of [unitary!] rings, show that the canonical map $R \to S^{-1}R$ is an epimorphism. For instance, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism of rings.

2. **Induced Homomorphism Theorem.**

$$\begin{array}{ccccccccccc} & & & & X & & & & & & \\ & & & & \Big\downarrow{\scriptstyle\varphi} & & & & & & \\ 0 & \longrightarrow & K & \xrightarrow{\alpha} & M & \xrightarrow{\zeta} & N & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & & & & & {\scriptstyle\psi}\Big\downarrow & & & & \\ & & & & & & Y & & & & \end{array}$$

Assume that the row above is exact.

a) Prove that φ "factors through K" (i.e. there exists $\varphi'\colon X \to K$ such that $\varphi = \alpha\varphi'$) if and only if $\zeta\varphi = 0$.

b) Prove that ψ "factors through C" (i.e. there exists $\psi'\colon C \to Y$ such that $\psi = \psi'\beta$) if and only if $\psi\zeta = 0$.

3. The Five Lemma. Hungerford, p. 180, Problem 12.

4. The Snake Lemma. Start with a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{\mu_1} & M_2 & \xrightarrow{\mu_2} & M_3 & \longrightarrow & 0 \\
 & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_3}\downarrow & & \\
0 & \longrightarrow & N_1 & \xrightarrow{\nu_1} & N_2 & \xrightarrow{\nu_2} & N_3 & &
\end{array}$$

Now for $i = 1, 2, 3$, let $K_i = \operatorname{Ker} \varphi_i$ and $L_i = \operatorname{Coker} \varphi_i$.

a) Prove that this yields a commutative diagram with exact rows and columns as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_1 & \xrightarrow{\kappa_1} & K_2 & \xrightarrow{\kappa_2} & K_3 & & \\
 & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow & & \\
 & & M_1 & \xrightarrow{\mu_1} & M_2 & \xrightarrow{\mu_2} & M_3 & \longrightarrow & 0 \\
 & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_3}\downarrow & & \\
0 & \longrightarrow & N_1 & \xrightarrow{\nu_1} & N_2 & \xrightarrow{\nu_2} & N_3 & & \\
 & & {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi_2}\downarrow & & {\scriptstyle\pi_3}\downarrow & & \\
 & & L_1 & \xrightarrow{\lambda_1} & L_2 & \xrightarrow{\lambda_2} & L_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

b) Prove that if μ_1 is monic then κ_1 is monic and if ν_2 is epic then λ_2 is epic.

c) Now define $\delta\colon K_3 \to L_1$ as follows:

$$\begin{array}{ccccc}
 & & & & k_3 \\
 & & & & \| \\
 & & m_2 & \xrightarrow{\mu_2} & k_3 \\
{\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_3}\downarrow \\
n_1 & \xrightarrow{\nu_1} & \varphi_2(m_2) & \xrightarrow{\nu_2} & 0 \\
{\scriptstyle\pi_1}\downarrow & & & & \\
\delta(k_3) & & & &
\end{array}$$

Let $k_3 \in K_3 \subseteq M_3$.

Choose m_2 such that $\mu_2(m_2) = k_3$.

Observe that $\varphi_2(m_2) = \nu_1(n_1)$ for a unique $n_1 \in N_1$.

Now define $\delta(k_3) = \pi_1(n_1) \in L_1$.

Prove that δ is a well defined homomorphism and that the resulting sequence $K_1 \to K_2 \to K_3 \xrightarrow{\delta} L_1 \to L_2 \to L_3$ is exact.

4.

$$\begin{array}{ccccc}
 & & k_2 & \xrightarrow{\kappa_2} & k_3 \\
 & & \| & & \| \\
m_1 & & m_2 & \xrightarrow{\mu_2} & k_3 \\
{\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_3}\downarrow \\
n_1 & \xrightarrow{\nu_1} & n_2 & \xrightarrow{\nu_2} & 0 \\
{\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi_2}\downarrow & & \\
\delta(k_3) & \xrightarrow{\lambda_1} & \nu_2(n_2) & &
\end{array}$$

(1) $\kappa_2(K_2) \subseteq \operatorname{Ker}\delta$: If $k_3 = \kappa_2(k_2)$ then $k_3 = \mu_2(k_2)$ so we may choose $m_2 = k_2$, in which case $\varphi_2(m_2) = 0$ and so $\delta(k_3) = 0$, so $k_3 \in \operatorname{Ker}\delta$.

(2) $\operatorname{Ker}\delta \subseteq \kappa_2(K_2)$: Conversely, if $\delta(k_3) = 0$, then $\pi_1(n_1) = 0$, so $(\exists m_1)\, n_1 = \varphi_1(m_1)$. Then $\varphi_2(m_2) = \nu_1(n_1) = \nu_1\varphi_1(m_1) = \varphi_2(\mu_1(m_1))$, so $m_2 - \mu_1(m_1) \in \operatorname{Ker}\varphi_2 = K_2$. But since $\mu_2(m_2 - \mu_1(m_1)) = \mu_2(m_2)$, we may replace m_2 with $m_2 - \mu_1(m_1)$. I.e. there is no loss of generality in this case in supposing that $m_2 \in K_2$. Then $k_3 = \mu_2(m_2) = \kappa_2(m_2) \in \kappa_2(K_2)$.

(3) $\delta(K_3) \subseteq \operatorname{Ker}\lambda_1$: If $k_3 \in K_3$, then $\lambda_1(\delta(k_3)) = \lambda_1\pi_1(n_1) = \pi_2\nu_1(n_1) = \pi_2\varphi_2(m_2) = 0$ so $\delta_1(k_3) \in \operatorname{Ker}\lambda_1$.

(4) $\operatorname{Ker}\lambda_1 \subseteq \delta(K_3)$: Conversely, suppose $\ell_1 \in \operatorname{Ker}\lambda_1$. We may write $\ell_1 = \pi_1(n_1)$ for some n_1. Then $0 = \lambda_1\pi_1(n_1) = \pi_2\nu_1(n_1)$ so that $\nu_1(n_1) \in \operatorname{Ker}\pi_2 = \varphi_2(M_2)$, say $\nu_1(n_1) = \varphi_2(m_2)$. Now $\varphi_3\mu_2(m_2) = \nu_2\varphi_2(m_2) = \nu_2\nu_1(n_1) = 0$, so $\mu_2(m_2) \in \operatorname{Ker}\varphi_3 = K_3$. But now we see that $\delta(\mu_2(m_2)) = \pi_1(n_1) = \ell_1$, so that $\ell_1 \in \delta(K_3)$.