MATH 612 **Due March 10, 1993**

1. Let P be an **artinian** projective module.

a) Prove that the following are equivalent:

(1) P is indecomposable.
(2) If M and N are proper submodules of P then $M + N \neq P$.
(3) P has a **unique** maximal proper submodule.

b) Prove that the unique maximal submodule in (3) above is JP, where $J = J(R)$.

2. **a)** Prove that if P is an indecomposable artinian projective module then P/JP is simple.

b) Prove that if S is a simple module over an **artinian** ring R then $S \approx P/JP$ for some indecomposable projective module P. Prove further that P is uniquely determined up to isomorphism by S.

Recall that if F and G are functors from a category $\mathcal{C}$ into a category $\mathcal{D}$ then a **natural transformation** from F to G is a collection of morphisms $\eta_X : F(X) \to G(X)$ for each object X in $\mathcal{C}$ such that for each morphism $\varphi \colon X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle \eta_X}\big\downarrow & & {\scriptstyle \eta_Y}\big\downarrow \\ G(X) & \xrightarrow{G(\varphi)} & G(Y). \end{array}$$

3. Let F be the identity functor on the category of left modules over a fixed ring R, i.e. $F(M) = M$ for all M and $F(\varphi) = \varphi$ for all $\varphi \in \mathrm{Hom}_R(M, N)$. Prove that the class of natural transformations from F to itself has a natural ring structure and is isomorphic to the center of R. (Note that it follows that this class is in fact a **set**.)

4. Let F be a functor from the category of left R-modules to the category of abelian groups, let M be a fixed R-module, and let $\mathrm{Nat}(\mathrm{Hom}_R(M, \_\_), F)$ be the set of natural transformations from $\mathrm{Hom}_R(M, \_\_)$ into F. Prove that the mapping $\tau \mapsto \tau_M(1_M)$ is an isomorphism between $\mathrm{Nat}(\mathrm{Hom}_R(M, \_\_), F)$ and $F(M)$.