1. a) Let A be an R-algebra and let G be an R-module. Recall that we consider $\mathrm{Hom}_R(A, G)$ as an A-module, where for $\varphi \in \mathrm{Hom}_R(A, G)$ and $a \in A$ we define $a\varphi$ by $(a\varphi)(x) = \varphi(xa)$. Prove that the isomorphism $\omega_M \colon \mathrm{Hom}_A(M, \mathrm{Hom}_R(A, G)) \to \mathrm{Hom}_R({}_RM, G)$ defined for A-modules M by $\omega_M(\alpha)(m) = \alpha(m)(1)$ is natural in with respect to the variable M, i.e. that whenever $\mu \colon M \to M'$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_A(M', \mathrm{Hom}_R(A, G)) & \longrightarrow & \mathrm{Hom}_A(M, \mathrm{Hom}_R(A, G)) \\ {\scriptstyle \omega_{M'}}\downarrow & & {\scriptstyle \omega_M}\downarrow \\ \mathrm{Hom}_R(M', G) & \xrightarrow{\mu^*} & \mathrm{Hom}_R(M, G) \end{array}$$

b) Prove in detail that if G is an injective R-module then $\mathrm{Hom}_R(A, G)$ is an injective A-module.

c) Let M be an A-module, G be an R-module, and $\gamma \colon M \to G$ an R-linear mapping. Define $\hat{\gamma} \colon M \to \mathrm{Hom}_R(A, G)$ by the following composition of maps: $M \xrightarrow{\approx} \mathrm{Hom}_A(A, M) \hookrightarrow \mathrm{Hom}_R(A, M) \xrightarrow{\gamma_*} \mathrm{Hom}_R(A, G)$, where the first map is the usual isomorphism and the second is the inclusion map. For $m \in M$ and $a \in A$, what is $\hat{\gamma}(m)(a)$? Prove that if γ is a monomorphism, then so is $\hat{\gamma}$.

2. Let R be a commutative ring and $A = R[X]$, the ring of polynomials in one variable with coefficients in R.

a) Prove that if V is an R-module then $\mathrm{Hom}_R(A, V) \approx \prod_0^\infty V$, where the A-module structure is given as follows: if $f \in \prod_0^\infty V$ then $X^n f$ is the sequence f shifted n spaces to the left; i.e. the k^{th} coordinate of $X^n f$ is the same as the $k + n^{\text{th}}$ coordinate of f.

b) Prove directly (without using theorems from class) that for any A-module M, $\mathrm{Hom}_A(M, \prod_0^\infty V) \approx \mathrm{Hom}_R(M, V)$.

c) Prove directly (without using problem 1) that if R is a field, then $\mathrm{Hom}_R(A, V)$ is always injective.

d) Let M be an A-module, let $\varphi \in \mathrm{Hom}_R(A, M)$, and let $f \in \prod_0^\infty M$ correspond to φ. Prove that φ is A-linear $\iff$ $(\forall n)\, f_n = X^n f_0$.

e) Let M be an A-module and define $\rho \colon M \to \prod_0^\infty M$ as the composition $M \xrightarrow{\approx} \mathrm{Hom}_A(A, M) \hookrightarrow \mathrm{Hom}_R(A, M) \xrightarrow{\approx} \prod_0^\infty M$, where the first map is the usual isomorphism. Prove that $\prod_0^\infty M$ is **not** (in most cases) an essential extension of $\rho(M)$.

f) Assuming that R is a field compute the injective envelope of $\rho(M)$ in $\prod_0^\infty M$.

The Symmetric Algebra. If R is a commutative ring and M is an R-module, define $T_R(M) = \bigoplus_0^\infty M^{\otimes n}$, where $M^{\otimes 0} = R$ and for $n > 1$, $M^{\otimes n} = M \otimes_R \cdots \otimes_R M \quad (n \text{ factors})$. With multiplication defined in the obvious way, this is called the **tensor algebra** for M over R. Now let I be the ideal generated by all elements of the form $m \otimes m' - m' \otimes m$ for $m \in M$.
(Note that, for instance,

$$\begin{aligned} m_1 \otimes m_2 \otimes m_3 &\equiv m_2 \otimes m_1 \otimes m_3 \\ &\equiv m_2 \otimes m_3 \otimes m_1 \\ &\equiv m_3 \otimes m_2 \otimes m_1 \pmod{I}.) \end{aligned}$$

The R-algebra $S_R(M) = T_R(M)/I$ is commutative and is called the **symmetric algebra** of M over R.

3. Prove that if F is a free R-module of rank n $(n < \infty)$ then $S_R(F) \approx R[X_1, \ldots, X_n]$.

4. Prove that $S_R(M)$ is characterized by the following universal property:

There is a homomorphism of R-modules $\delta\colon M \to S_R(M)$ and if A is any commutative R-algebra and $\varphi\colon M \to A$ is a homomorphism of R-modules, then there is a unique homomorphism μ of R-algebras such that the following diagram commutes:

$$\begin{array}{ccc} M & & \\ {\scriptstyle\delta}\big\downarrow & & \\ S_R(M) & \xrightarrow{\ \mu\ } & A. \end{array}$$