FACTS ON SEMI-SIMPLE ARTINIAN RINGS

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Theorem [Wedderburn-Artin]. Let R be a ring. The following are equivalent:

(1) R is a direct sum of **mutually isomorphic** minimal left ideals.
(2) R is a direct sum of **mutually isomorphic** minimal right ideals.
(3) R is a simple left (right) artinian ring.
(4) R is isomorphic to the ring of $n \times n$ matrices over some skew-field D.

Theorem. Let R be a ring. The following are equivalent:

(1) R is a finite direct sum of minimal left ideals (not necessarily mutually isomorphic).
(2) R is a direct sum of minimal right ideals.
(3) R is a finite product of simple left (right) artinian rings.
(4) R is left (right) artinian and $J(R) = 0$.

A ring satisfying the equivalent conditions above is called (artinian) **semi-simple**.

Corollary. If R is left artinian then $R/J(R)$ is semi-simple.

PROOF: It suffices to prove that $J(R/J(R)) = 0$. In fact, the maximal left ideals of $R/J(R)$ are the images of the maximal left ideals of R containing $J(R)$. But all maximal left ideals of R contain $J(R)$. Thus $J(R/J(R))$ is the image of the intersection of all the maximal left ideals of R, i. e. the image of $J(R)$, which is the 0 ideal in $R/J(R)$.

Proposition. If a left ideal N in a ring R consists entirely of nilpotent elements, then $N \subseteq J(R)$.

Proposition. In a **commutative** ring R, the set of all nilpotent elements is an ideal.

Proposition. Let R be a left artinian ring and let J be its Jacobson radical. Then $J^k = 0$ for some $k \geq 1$. In particular, J consists of nilpotent elements.

PROOF: By the minimum condition, among the ideals of the form J^k there exists a smallest one. Since $J^{k+1} \subseteq J^k$ (WHY?), we see that $J^k J = J^{k+1} = J^k$. Thus if $J^k \neq 0$ then there exist left ideals $L \subseteq J$ such that $J^k L \neq 0$. Since R has minimum condition we can choose a smallest such left ideal L. Then $J^k \ell \neq 0$ for some $\ell \in L$. Now $J^k \ell \subseteq L$ (WHY?) and $J^k (J^k \ell) = J^{2k} \ell = J^k \ell \neq 0$. Thus by the minimality of L, $J^k \ell = L$. In particular, $\ell \in J^k \ell$ so $\ell = r\ell$ for some $r \in J^k \subseteq J$, so $(1 - r)\ell = 0$. But since $J = J(R)$, $1 - r$ is invertible. This implies $\ell = 0$ a CONTRADICTION. Thus $J^k = 0$.

In particular, if $r \in J$ then $r^k = 0$. Thus J consists of nilpotent elements. ☑

We can now say that

In a left artinian ring R, the Jacobson radical is the largest left ideal consisting completely of nilpotent elements.

Proposition. Let R be a ring and $J = J(R)$. Suppose that R/J is left artinian. Let M be an R-module such that $JM = 0$. Then M is a (possibly infinite) direct sum of simple R-modules.

PROOF: Since $JM = 0$, we can think of M as an R/J-module by setting $(r + J)m = rm$. (Note that this is well defined.) Therefore there is no loss of generality in assuming that $J = 0$. In this case, R is artinian semi-simple and therefore, as seen above, is a finite direct sum of simple left ideals. It follows that M is generated by the set of submodules of the form Lm where L is a simple left ideal and $m \in M$. Furthermore, by Schur's Lemma either $Lm = 0$ or $Lm \approx L$, so Lm is a simple submodule of M.

The completion of the reasoning is given in Hungerford, Theorem 3.6, p. 437. ☑

Corollary. Let R be a ring and $J = J(R)$. Suppose that R/J is left artinian. For all $i \geq 1$, J^i/J^{i+1} is a finite direct sum of simple modules and in particular has finite length.

PROOF: Since $J^i \subseteq R$, then J^i is an artinian R-module and therefore J^i/J^{i+1} is artinian. Therefore J^i/J^{i+1} is a finite direct sum of indecomposable modules. But it follows from the preceding proposition that an indecomposable submodule of J^i/J^{i+1} must be simple. ☑

Theorem. A left artinian ring R is also right artinian and is left and right noetherian, and thus has finite length as both a left and right R-module.

PROOF: Let $J = J(R)$. We prove by induction for for all $i \geq 1$, R/J^i has finite length both as a left and right R-module. Since $J^k = 0$ for some k, this will prove the theorem. For $i = 1$ this follows from the Wedderburn-Artin Theorem since R/J is a semi-simple artinian ring and is thus a finite direct sum of simple left (right) R-modules.

Thus induction step follows by considering the short exact sequence

$$0 \to J^i/J^{i+1} \to R/J^{i+1} \to R/J^i \to 0$$

and using the preceding corollary. ☑