A Proof of the Wedderburn-Artin Theorem

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Theorem. Let R be a ring satisfying the descending chain condition on left ideals and with no two-sided ideals except (0) and R. Then there exists a positive integer n such that for any minimal non-trivial left ideal L of R the following hold:

(1) R is isomorphic to a direct sum of n copies of L.
(2) If $D = \operatorname{End}_R L$ then D is a skew field and L is an n-dimensional vector space over D.
(3) $R \approx \operatorname{End}_D L$.

Schur's Lemma. If L is a simple R-module then $\operatorname{End}_R L$ is a skew field.

PROOF: Since $\operatorname{End}_R L$ is a ring, it suffices to see that every non-zero endomorphism $\varphi \in \operatorname{End}_R L$ is invertible, i. e. is an automorphism of L. Now $\operatorname{Ker}\varphi$ is a submodule of of L and if $\varphi \neq 0$ then $\operatorname{Ker}\varphi \neq L$ so $\operatorname{Ker}\varphi = 0$ since L is simple. Thus φ is monic. Likewise $\varphi(L)$ is a submodule of L and if $\varphi \neq 0$ then $\varphi(L) \neq 0$ so $\varphi(L) = L$ since L is simple. Thus φ is surjective, so φ is an automorphism of L and thus is invertible in $\operatorname{End}_R L$. ☑

Lemma 1. If M is a cyclic left R-module satisfying the descending chain condition on submodules and generated by a set of isomorphic copies of a single R-module S, then M is isomorphic to a direct sum of finitely many copies of S.

PROOF: Let m generate M. By Zorn's Lemma there exists a submodule $M' \subseteq M$ maximal with respect to the property that $m \notin M'$. Then M/M' is a simple R-module since any proper submodule would have the form M''/M' where $M' \subseteq M'' \subseteq M$ and $m \notin M''$ (since m generates M), so that $M'' = M'$ by the maximality of M'. Furthermore since M is generated by isomorphic copies of S there must exist a submodule S' of M such that $S' \approx S$ and $S' \not\subseteq M'$. The restriction of the quotient map $\varphi\colon M \to M/M'$ to S' is non-trivial, so since S' and M/M' are simple we conclude that $\varphi(S') = M/M'$. This says that $M = M' + S'$. But $M' \cap S' \subsetneq S'$ and S' is simple, so $M' \cap S' = 0$ and thus $M = M' \oplus S'$. Now M' is a homomorphic image of M, hence is cyclic, so the same argument shows that there is a direct sum decomposition $M' = M'' \oplus S''$ for some simple S''. By the descending chain condition, the chain $M \supsetneq M' \supsetneq M'' \supsetneq \dots$ must eventually reach 0 and $M = S' \oplus S'' \oplus \cdots \oplus S^{(n)}$. ☑

Remark. A more careful use of Zorn's Lemma shows that Lemma 1 holds even if M is not cyclic and does not satisfy the descending chain condition, except that in the latter case M may be a direct sum of an infinite number of copies of S. [Lang, Chapter XVII §2, p. 441].

Lemma 2. Let R be a ring and L be a left R-module and $D = \mathrm{End}_R L$. If $R \approx L^n$ for some n then L is a free D-module having a basis of n elements and $R \approx \mathrm{End}_D L$.

PROOF: Let $\theta\colon L^n \to R$ be an R-linear isomorphism and let

$$\theta^{-1}(1) = (u_1, \ldots, u_n) \in L^n.$$

We claim that $u_1, \ldots, u_n$ is a D-basis for L. Fix $\ell \in L$. For $1 \le k \le n$, define $\delta_k\colon L \to L$ by

$$\delta_k(x) = \theta(0, \ldots, x, \ldots, 0)\ell$$

(where only the k^{th} coordinate is non-zero). We see that for $r \in R$, $\delta_k(rx) = r\delta_k(x)$ so that $\delta_k \in \mathrm{End}_R L = D$. Furthermore

$$\ell = 1\ell = \theta(u_1, \ldots, u_n)\ell = \sum \delta_k(u_k).$$

Thus $\ell \in \sum Du_k$. Since this can be done for every $\ell \in L$, it follows that $u_1, \ldots, u_n$ span L as a D-module.

Now if $m = (x_1, \ldots, x_n) \in L^n$ and $r = \theta(m)$ then

$$(x_1, \ldots, x_n) = m = \theta^{-1}(r \cdot 1) = r\theta^{-1}(1) = r(u_1, \ldots, u_n) \in L^n,$$

i.e. $x_i = ru_i$ for all i. In particular, for any given i and x_i, applying this where $x_j = 0$ for all $j \neq i$ shows that there exists $r \in R$ such that

$$ru_i = x_i \quad \text{and} \quad ru_j = 0 \quad \text{for} \quad j \neq i.$$

It follows that the $u_1, \ldots, u_n$ are linearly independent over D. In fact, consider a linear relation $\delta_1 u_1 + \cdots + \delta_n u_n = 0$. Fix i and $x_i \in L$ and let r be as above. Then $0 = r(\delta_1 u_1 + \ldots \delta_n u_n) = \sum \delta_j(ru_j) = \delta_i(x_i)$. Since x_i can be an arbitrary element of L, this shows that $\delta_i = 0$. Thus $u_1, \ldots, u_n$ are linearly independent over D and since we have seen that they span L as a D-module, they are a D-basis for L. Hence L is a free D-module with rank n.

Now let $\varphi \in \mathrm{End}_D L$. Choosing $x_i = \varphi(u_i)$ for all i, the above calculation shows that there exists $r \in R$ such that $\varphi(u_i) = ru_i$ for all i. Since $u_1, \ldots, u_n$ span L as a D-module and left multiplication by r is D-linear, it follows that $\varphi(x) = rx$ for all $x \in L$, i.e. φ is the same as left multiplication by r. Furthermore r is unique since if $rx = r'x$ for all $x \in L$ then $\theta(r) = r\theta(1) = (ru_1, \ldots, ru_n) = (r'u_1, \ldots, r'u_n) = r'\theta(1) = \theta(r')$ and so $r = r'$ since θ is an isomorphism. Thus every $\varphi \in \mathrm{End}_D L$ is given by left multiplication by a unique $r \in R$ so that $\mathrm{End}_D L \approx R$. ☑

Proof of Theorem. If L is a minimal left ideal in R (whose existence is guaranteed by the descending chain condition) then the left ideal in R generated by Lr for all $r \in R$ is quickly seen to be a non-trivial two-sided ideal, hence by hypothesis must be R itself. Since L is a simple R-module, the R-linear map given by $x \mapsto xr$ is an isomorphism from L to Lr [Schur's Lemma]. Thus R is generated by isomorphic copies of L so by Lemma 1, $R \approx L^n$ for some n. Clearly n is the length of R as a left R-module, hence is unique. By Schur's Lemma, $D = \mathrm{End}_R L$ is a skew field and by Lemma 2, $R \approx \mathrm{End}_D L$ and L has dimension n over D. ☑