

**WHAT I KNOW ABOUT GROTHENDIECK RINGS FOR TORSION FREE
CATEGORIES
(AND ONE OR TWO THINGS I'D LIKE TO KNOW)**

E. L. LADY

(Maybe around 1987)

It seems to me that understanding the ring structure induced by the tensor product on the Grothendieck group of quasi-isomorphism classes of finite rank torsion free groups is a worthwhile and intriguing goal. Apparently no one else feels the same way, since nobody else seems to be working on the problem, a situation which certainly has its advantages for me but which also seems just a little bit sad. So I thought it might help inspire and/or encourage a few others if I presented a summary of what I know at this point, including the main results from my two papers on the subject along with some theorems that have been sitting down in the basement gathering dust because I haven't yet managed to find a suitable family for them to belong to.

In keeping with my usual custom, we will look at finite rank torsion free modules over a dedekind domain W . We consider the category where the morphisms are quasi-homomorphisms, and let \mathbf{K} be the Krull-Schmidt-Grothendieck ring for this category. Thus elements of \mathbf{K} are differences $[G] - [H]$ of quasi-isomorphism classes of finite rank torsion free W -modules, and addition corresponds to quasi-direct sums and multiplication to tensor products: $[G][H] = [G \otimes H]$. Because of the well known Krull-Schmidt Property for quasi-direct decompositions, \mathbf{K} is the free abelian group on the set of quasi-isomorphism classes of strongly indecomposable modules. For $x \in \mathbf{K}$, we let $\text{Support } x$ denote the set of classes $[G]$ of strongly indecomposable modules which occur with non-zero coefficient in the expression of x as a linear combination of these basis elements.

All modules that occur are understood to be finite rank torsion free. \mathbb{Z} will denote the ring of integers and \mathbb{Q} the field of rational numbers.

1. Subrings for Subcategories. We will be interested in various full subcategories of the category where the objects are finite rank torsion free W -modules and the morphisms are quasi-homomorphisms. By a **suitable subcategory** is meant a full subcategory \mathcal{C} which is closed under tensor products and quasi-direct sums and summands. The corresponding subgroup $\mathbf{K}(\mathcal{C})$ of \mathbf{K} is then closed under multiplication; it is a subring of \mathbf{K} if and only if W is in \mathcal{C} .

Definition. A **generalized Butler module** is a module G such that for some finite integral extension W' of W , $W' \otimes G$ is a Butler W' -module.

The category of **quotient divisible generalized Butler modules** will be denoted by $QDGB$; these modules are characterized in [Sem Theorem 6.25]. (A Butler module is quotient divisible if and

only if its type set contains only idempotent types.) $\mathbf{K}(\mathcal{QDGB})$ was studied in some detail in [Groth]. Note that in particular, all domains are in \mathcal{QDGB} . (By a **domain** we mean a finite rank torsion free integral domain over W . Since a domain is quasi-equal to its integral closure, we can usually assume without loss of generality that D is a dedekind domain.) We let \mathcal{DSD} denote the category of quasi-direct sums of domains.

We let \mathcal{LF} denote the category of locally free W -modules.

We define W -rank G (as contrasted to $\text{rank}_W G$) to be the rank of a maximal projective summand of G . W -rank extends to a ring morphism $\mathbf{K} \rightarrow \mathbb{Z}$, and for $x \in \mathbf{K}$, W -rank $x \neq 0$ if and only if $1 = [W] \in \text{Support } x$.

If R and S are rings, we say that R is a **retract** of S if R is a subring of S and there is a ring morphism $\rho: S \rightarrow R$ such that ρ restricts to the identity map on R . Equivalently, $S = R \oplus I$, where I is an ideal in S .

1.1 Retracts. (1) \mathbb{Z} is a retract of \mathbf{K} .

(2) If $\text{char } W = 0$ then $\mathbf{K}(\mathcal{LF})$ is a retract of \mathbf{K} .

(3) $\mathbf{K}(\mathcal{DSD})$ is a retract of $\mathbf{K}(\mathcal{QDGB})$.

PROOF: (1) We identify \mathbb{Z} with the cyclic group generated by $1 = [W] \in \mathbf{K}$, which is the subring of \mathbf{K} corresponding to the category of projective W -modules. Then the ring morphism W -rank (or, for that matter D -rank, for any domain D) is a retraction of \mathbf{K} onto \mathbb{Z} .

(2) By [WD&ROQ, Theorem 2.7], the subgroup of \mathbf{K} corresponding to the category whose objects are those W -modules G having no locally free quasi-summands with non-trivial trace is an ideal. If we call this ideal I , then in characteristic 0, $\mathbf{K} = \mathbf{K}(\mathcal{LF}) \oplus I$. (In characteristic 0, every module has non-trivial trace.)

(3) $\mathbf{K}(\mathcal{QDGB}) = \mathbf{K}(\mathcal{DSD}) \oplus \mathbf{N}$, where \mathbf{N} is its nil radical [Groth, Theorem 2]. \square

1.2. Integral Closure. (1) $[G]$ is integral over \mathbb{Z} if and only if G is in \mathcal{QDGB} .

(2) $x \in \mathbf{K}$ is integral over \mathbb{Z} if and only if $x = y + n$ where $y \in \mathbf{K}(\mathcal{QDGB})$ and n is nilpotent. In fact, we can choose $y \in \mathbf{K}(\mathcal{DSD})$.

(3) More generally, let \mathcal{C} be a suitable subcategory containing W and let \mathcal{C}_e be the category whose objects are quasi-direct sums of modules of the form $D \otimes G$, with $G \in \mathcal{C}$ and D a domain. (This includes, in particular, modules of the form $S^{-1}G$ for multiplicative sets S in W .) Then $\mathbf{K}(\mathcal{C}_e)$ is the subring generated by $\mathbf{K}(\mathcal{C}) + \mathbf{K}(\mathcal{DSD})$, and $\mathbf{K}(\mathcal{C}_e)$ is integral over $\mathbf{K}(\mathcal{C})$.

PROOF: (1) and (2) are in [IncCl]. (3) is the fairly obvious. \square

We say that a ring S is a **free extension** of a ring R if R is a subring of S and S is a free R -module.

The following result is just an elaborate restatement of the standard way of classifying rank-one modules by height sequences. It's interesting, though, because it shows that $\mathbf{K}(\mathcal{LF})$ is a free extension of a known ring which is determined only by the cardinality of the set of prime ideals in W .

1.3. Locally Free Complete Decomposable Modules. Let \mathcal{LF} be the category of locally free modules and let \mathcal{LFCD} be the category of locally free completely decomposable modules. (Note: A rank-one module is locally free if and only if its type is locally trivial (i. e. the associated height sequence has no infinities). Thus \mathcal{LFCD} consists of all direct sums of rank-one modules with locally trivial type.) Let I be the set of non-zero prime ideals in W , let $\Sigma = \mathbb{Z}^{(I)}$ and $\Pi = \mathbb{Z}^I$. Then $\mathbf{K}(\mathcal{LFCD})$ is isomorphic to the subring of the integral group ring of Π/Σ generated by those cosets represented by non-negative sequences, and $\mathbf{K}(\mathcal{LF})$ is a free extension of $\mathbf{K}(\mathcal{LFCD})$. A basis for $\mathbf{K}(\mathcal{LF})$ over $\mathbf{K}(\mathcal{LFCD})$ consists of those elements $[G]$ where G is a strongly indecomposable locally free W -module whose inner type is $\mathbf{t}(W)$.

PROOF: $\mathbf{K}(\mathcal{LFCD})$ is the free abelian group on the elements $[A]$, with $[A]$ a locally free rank-one module. Such modules are determined up to quasi-isomorphism by equivalence classes of finite-valued height sequences, i. e. by non-negative elements in Π/Σ . Furthermore, the tensor product of two such modules corresponds to the addition of the height sequences. Thus $\mathbf{K}(\mathcal{LFCD})$ is a subring of the group ring as claimed. Now if $[G]$ is a locally free module, let A be a rank-one module such that $\mathbf{IT}(G) = \mathbf{t}(A) = [A]$ (according to the convention of defining types as quasi-isomorphism classes of rank-one modules). Then $G \approx A \otimes G_0$, where $G_0 = \text{Hom}_W(A, G)$, and so $[G] = [A][G_0]$. Thus the elements $[G_0]$ such that $[G_0]$ is locally free and strongly indecomposable and $\mathbf{IT}(G) = 1$ form a generating set for $\mathbf{K}(\mathcal{LF})$ over $\mathbf{K}(\mathcal{LFCD})$. That this set is linearly independent over $\mathbf{K}(\mathcal{LFCD})$ follows easily from the Krull-Schmidt Property for the category of torsion free W -modules under quasi-homomorphisms and from Warfield Duality. Hence $\mathbf{K}(\mathcal{LF})$ is a free extension of $\mathbf{K}(\mathcal{LFCD})$. \square

1.4. Similarity. In the embedding of $\mathbf{K}(\mathcal{LFCD})$ into the group ring $\mathbb{Z}[\Pi/\Sigma]$, the rank-one modules A are mapped to the group elements. Recall that the **augmentation ideal** of the group ring consists of those elements $\sum n_i g_i$ such that $\sum n_i = 0$. It is generated by those elements of the form $g - 1$ for g in the group (i. e. $g \in \Pi/\Sigma$). Let AUG be the intersection of $\mathbf{K}(\mathcal{LFCD})$ with the augmentation ideal of the group ring $\mathbb{Z}[\Pi/\Sigma]$. Then we see that

$$\text{AUG} = \{x \in \mathbf{K}(\mathcal{LFCD}) \mid \text{rank } x = 0\}$$

and AUG is generated as a group by elements of the form $[A] - [W]$, where A is rank-one and locally free. Then AUG is a prime ideal in $\mathbf{K}(\mathcal{LFCD})$ and $\mathbf{K}(\mathcal{LFCD})/\text{AUG} \approx \mathbb{Z}$. Let \mathbf{K}_{Sim} be the quotient of \mathbf{K} modulo the ideal in \mathbf{K} generated by AUG . We say that $x, y \in \mathbf{K}$ are **similar** if they yield the same residue class in \mathbf{K}_{Sim} . Two strongly indecomposable modules G and H are similar if and only if there exist locally free rank-one modules A and B with $A \otimes G \approx B \otimes H$. This says that G and H are structurally very much like each other, except for the types of their elements. The study of \mathbf{K}_{Sim}

seems a promising way to gain further understanding of \mathbf{K} , because it focuses attention on the deeper structure of modules and away from differences which merely have to do with the types of elements.

2. Idempotents. Obviously if $[A]$ is a rank-one module with idempotent type, then $[A]$ and $1 - [A]$ are idempotents in \mathbf{K} .

Let \mathcal{DSD} be the full subcategory of \mathcal{C} whose objects are quasi-direct sums of domains. By [Sem, Proposition 6.9 and Lemma 6.19], \mathcal{DSD} is a suitable subcategory (i. e. it's closed under tensor products, quasi-direct sums, and quasi-direct summands, so it corresponds to a subring of \mathbf{K}).

For a domain D and any W -module G , we define

$$D\text{-rank } G = \text{rank}_D \text{Hom}(G, D).$$

D -rank G also equals the rank of a maximal D -projective summand of the D -module $D \otimes G$. (This depends on the fact that we may assume, without loss of generality, that D is a dedekind domain.) The set of strongly indecomposable domains can be partially ordered by setting $[D_1] \leq [D_2]$ if and only if D_1 is isomorphic to a subring of a finite integral extension of D_2 (see [IncCl]).

If D is a domain with rank greater than one, then clearly $D \otimes D$ cannot be isomorphic to D , so $[D]$ cannot be an idempotent in \mathbf{K} . However the multiplication map $D \otimes D \rightarrow D$ splits, so that D is isomorphic to a summand of $D \otimes D$, and in fact, if D is strongly indecomposable, then $D \otimes D$ breaks up into a quasi-direct sum of integral domains, exactly one of which is quasi-isomorphic to D itself [IntCl]. We can use this to construct an idempotent corresponding to D . In fact, the next result shows that there is a one-to-one correspondance between the primitive idempotents in \mathbf{K} and the strongly indecomposable domains.

2.1. (1) $\mathbf{K}(\mathcal{DSD})$ is the subring of \mathbf{K} generated by the idempotents.

(2) An element $x \in \mathbf{K}(\mathcal{DSD})$ is an idempotent if and only if for each domain D , D -rank x is either 0 or 1.

(3) For each strongly indecomposable domain D there is a unique corresponding idempotent e_D such that for all domains D' ,

$$D'\text{-rank } e_D \neq 0 \iff [D'] \geq [D].$$

These idempotents generate the boolean algebra of idempotents in \mathbf{K} and therefore also generate $\mathbf{K}(\mathcal{DSD})$ as an abelian group.

PROOF: (1) [IntCl].

(2) By [Groth, Property C], for $x \in \mathbf{K}(\mathcal{DSD})$, $x^2 = x$ if and only if $(D\text{-rank } x)^2 = D\text{-rank } x^2 = D\text{-rank } x$ for all domains D , and this is equivalent to the condition that D -rank x is either 0 or 1 for each domain D .

(3) $\mathbb{Q} \otimes \mathbf{K}(\mathcal{DSD})$ is generated over \mathbb{Q} by idempotents, hence is von Neumann regular, so that every principal ideal is generated by an idempotent. Thus $[D] = ue_D$ for some idempotent e_D and some unit u of $\mathbb{Q} \otimes \mathbf{K}(\mathcal{DSD})$. Now e_D is actually in $\mathbf{K}(\mathcal{DSD})$ and D' -rank $u \neq 0$ (since it's a unit), so D' -rank $e_D \neq 0$ if and only if D' -rank $D \neq 0$, and this is true if and only if $[D'] \geq [D]$. \square

3. Prime Ideals. In 1.3 we saw that $\mathbf{K}(\mathcal{LFCD})$ is isomorphic to a subring of the integral group ring of Π/Σ where $\Pi = \mathbb{Z}^I$, where I is the set of primes in W , and $\Sigma = \mathbb{Z}^{(I)}$. Unless W is semi-local, Π/Σ is an infinite rank torsion free group, and it is known (and easily proved) that its integral group ring contains infinitely long chains of prime ideals. These give rise to infinitely long prime ideal chains in \mathbf{K} . I don't know whether such chains exist in \mathbf{K}_{Sim} .

The idea in part (2) of the following proposition is taken from Gilman and Jerrison's classic *Rings of Continuous Functions*.

3.1. Domains and Ultrafilters Thereof. (1) To each strongly indecomposable domain D there corresponds a prime ideal $P_D = \text{Ker}(D\text{-rank})$.

(2) Let S be the set of quasi-isomorphism classes of strongly indecomposable domains and for $x \in \mathbf{K}$ let

$$Z(x) = \{D \in S \mid D\text{-rank } x = 0\} = \{D \in S \mid x \in P_D\}.$$

Then the family of sets

$$\mathcal{B} = \{Z(x) \mid x \in \mathbf{K}\}$$

is a boolean algebra of sets and for each ultrafilter \mathcal{U} in \mathcal{B} there corresponds a prime ideal

$$P_{\mathcal{U}} = \{x \in \mathbf{K} \mid Z(x) \in \mathcal{U}\}.$$

(3) Every prime ideal P in \mathbf{K} containing $\bigcap P_D$ has the form $P_{\mathcal{U}}$ as in (2) and $\mathbf{K}/P \approx \mathbb{Z}$ = the ring of integers.

(4) Are there any other prime ideals P in \mathbf{K} with $\mathbf{K}/P \approx \mathbb{Z}$?

PROOF: (1) Obvious, and also a special case of (2).

(2) \mathcal{B} is closed under finite unions and intersections since $Z(x) \cup Z(y) = Z(xy)$ and $Z(x) \cap Z(y) = Z(x^2 + y^2)$. Now if $x = [G] - [H]$, then for any D , $-D\text{-rank } H \leq D\text{-rank } x \leq D\text{-rank } G$. Thus for given x there are only finitely many integers m such that $Z(x - m) \neq \emptyset$ and these form a finite partition of S . It follows that \mathcal{B} is also closed under complementation. And if \mathcal{U} is an ultrafilter in \mathcal{B} , it is routine to see that $P_{\mathcal{U}}$ is a prime ideal in \mathbf{K} .

(3) If $x \in \mathbf{K}$ then $D \in Z(x - n)$ if and only if $x - n \in P_D$. Thus $\prod_n (x - n) \in \bigcap P_D$, where the product here is taken over the set of integers n such that $Z(x - n) \neq \emptyset$. As shown in the preceding paragraph, this is a finite set of integers. Then if P is a prime ideal containing $\bigcap P_D$ and $x \in \mathbf{K}$, then $\prod (x - n) \in \bigcap P_D \subseteq P$, and hence $x - n \in P$ for some integer n . This shows that $\mathbf{K}/P \approx \mathbb{Z}$.

Note also that if $x \in P$ then $x - n \notin P$ for $n \neq 0$ because $n \notin P$. (By assumption, P is pure.) Hence there is at least one domain D such that $\prod_{n \neq 0} (x - n) \notin P_D$ (otherwise $\prod_{n \neq 0} (x - n) \in \bigcap P_D \subseteq P$). Since P_D is prime and $\prod (x - n) \in P_D$, thus $x \in P_D$, so that $D \in Z(x)$. This shows that $Z(x) \neq \emptyset$. It is now easily seen that $\mathcal{U} = \{Z(x) \mid x \in P\}$ is an ultrafilter in \mathcal{B} and clearly $P \subseteq P_{\mathcal{U}}$. Since $K/P \approx \mathbb{Z}$, P is a maximal pure ideal in \mathbf{K} , so $P = P_{\mathcal{U}}$. \square

Note that for an integer n ,

$$Z(x - n) = \{D \mid D\text{-rank } x = n\}.$$

Note that every domain D corresponds to an ultrafilter $\mathcal{U} = \{B \in \mathcal{B} \mid D \in B\}$ and then $P_D = P_{\mathcal{U}}$. Ultrafilters of this form are called **principal ultrafilters**. An ultrafilter is principal if and only if it contains at least one finite set.

Prime ideals P in \mathbf{K} such that $\mathbf{K}/P \approx \mathbb{Z}$ are important because each such prime ideal corresponds to a **rank function**: an integer-valued function defined on finite rank torsion free W -modules which is additive over direct sums and multiplicative with respect to tensor products. In respect to primes of the form $P_{\mathcal{U}}$, this rank function is defined by setting \mathcal{U} -rank x to be D -rank x , where D is a domain in any sufficiently small set belonging to \mathcal{U} . This is based on the fact that for sufficiently small $B \in \mathcal{U}$, D -rank x is constant over B . (The value of D -rank x will then be independent of B because of the fact that \mathcal{U} is closed under intersections.) To see that this is true, consider any set $B' \in \mathcal{U}$. The sets $\{D \in B' \mid D\text{-rank } x = m\}$ (m ranging over all integers) form a finite partition of B' . Since \mathcal{U} is an ultra-filter, exactly one set B in this partition will belong to \mathcal{U} . Then D -rank $x = m$ for all D belonging to any subset of B .

If Q is the quotient field of W , then for any G , Q -rank $G = \text{rank } G$. Thus P_Q consists of those elements $[G] - [H]$ in \mathbf{K} such that $\text{rank } G = \text{rank } H$.

3.2. P_Q is the principal ideal generated by the idempotent $1 - [Q]$ and is a minimal prime ideal in \mathbf{K} .

PROOF: If $x = [G] - [H] \in P_Q$ then $\text{rank } G = \text{rank } H$, so $[Q]x = 0$ since $Q \otimes G \approx Q \otimes H$. Thus $x = (1 - [Q])x$, showing that $1 - [Q]$ generates P_Q .

Furthermore, this shows that $[Q]P_Q = 0$. Now if P' is a prime ideal and $P' \subseteq P_Q$, then $0 = [Q]P_Q \subseteq P'$ and $[Q] \notin P'$ (since $[Q] \notin P_Q$), so $P_Q \subseteq P'$ since P' is by assumption a prime ideal. This shows that P_Q is a minimal prime ideal. \square

This sort of logic can be applied in a lot of situations. For instance, for a given prime p in W , let I be the ideal consisting of all elements of the form $[G] - [H]$, with G and H both p -divisible. This is the principal ideal generated by $p^{-\infty}W$. Since $([W_p] - [Q])[p^{-\infty}W] = 0$, it follows that $([W_p] - [Q])I = 0$. Consequently if P is a prime ideal in \mathbf{K} and there is at least one pair of p -divisible modules G and H such that $[G] - [H] \notin P$, then P contains $[W_p] - [Q]$.

One gets a prime ideal P_D for every dedekind domain D , even if D does not have finite rank. If we choose $D = \hat{W}_p$ (the p -adic completion of W), for instance, then the corresponding rank function is simply p -rank. I suspect, however, that allowing infinite-rank D doesn't really give much that's new, because I suspect that every such P_D is the same as P_U for some ultrafilter U .

For each prime ideal P such that $\mathbf{K}/P \approx \mathbb{Z}$ and each prime number p , there is a maximal ideal P' above P , namely the inverse image of the prime ideal (p) in \mathbb{Z} with respect to the homomorphism $\mathbf{K} \rightarrow \mathbb{Z}$ whose kernel is P . These prime ideals have only nuisance value, and in the interest of clearing them away it is probably easier to consider the ring $\mathbb{Q} \otimes \mathbf{K}$, where \mathbb{Q} is the field of rational numbers.

3.3. Quotient Divisible Generalized Butler Modules. (1) $\mathbb{Q} \otimes \mathbf{K}(\mathcal{QDGB})$ has Krull dimension 0, i. e. every prime ideal is both maximal and minimal. In fact, the prime ideals are precisely the contractions of the primes P_U (and P_D) in 3.1.

(2) If P is a minimal prime ideal in $\mathbf{K}(\mathcal{QDGB})$, then $\mathbf{K}(\mathcal{QDGB})/P \approx \mathbb{Z}$.

PROOF: $\mathbf{K}(\mathcal{QDGB}) = \mathbf{K}(\mathcal{DSD}) \oplus \mathbf{N}$, where \mathbf{N} is the nilradical [Groth]. Thus prime ideals in $\mathbf{K}(\mathcal{QDGB})$ correspond to primes in $\mathbf{K}(\mathcal{DSD})$. But $\mathbf{K}(\mathcal{DSD})$ is generated by idempotents, so (2) follows, and we see that $\mathbb{Q} \otimes \mathbf{K}(\mathcal{DSD})$ is von Neumann regular, so (1) follows. \square

4. Nil Radical. Let \mathbf{N} be the nil radical of \mathbf{K} . Since \mathbf{K} is commutative, \mathbf{N} consists precisely of the nilpotent elements and is the intersection of all the prime ideals. Clearly a necessary condition for $[G] - [H]$ to be nilpotent is that D -rank $G = D$ -rank H for all domains D . This condition is sufficient if G and H are quotient divisible generalized Butler modules, but it is clear from 3.1 that it is not sufficient in general. (There are easy counter-examples with G and H both being rank-one modules with non-idempotent type.)

5. Units.

5.1. (1) If u is a unit, then D -rank $u = \pm 1$ for all domains D .

(2) Conversely, every $u \in \mathbf{K}(\mathcal{QDGB})$ with this property is a unit.

(3) That's all, folks!

PROOF: (1) Clear.

(2) Replacing u by u^2 , we may assume that in fact for all D , D -rank $u = 1$. Then by [Groth, Property C], $1 - u$ is nilpotent, so $u = 1 - (1 - u)$ is invertible. \square

6. Zero Divisors and Regular Elements. Since \mathbf{K} has an abundance of idempotents, it has lots of zero divisors. A regular element in \mathbf{K} is, of course, one which is not a zero divisor.

The most obvious example of a zero divisor in \mathbf{K} is the element $[Q]$, where Q is the quotient field of W . If x is any element of \mathbf{K} with $\text{rank } x = 0$, then $[Q]x = 0$. This is because $x = [G] - [H]$ with $\text{rank } G = \text{rank } H$, and so $Q \otimes G = Q \otimes H$.

More generally, we have the following proposition:

- 6.1.** (1) If G and H are p -divisible for the same prime ideal p in W , then $[G] - [H]$ is a zero divisor.
 (2) If A is a rank-one module, then $[A]$ is a regular element if and only if it has locally trivial type.

PROOF: (1) This has already been seen in Section 3. If G and H are p -divisible, then $G_p = QG$ and $H_p = QH$, so that $([W_p] - [Q])[G] = ([W_p] - [Q])[H] = 0$.

(2) To say that A has locally trivial type is to say that it is not p -divisible for any prime p of W . Thus by (1), if $\mathfrak{t}(A)$ is not locally trivial then $[A]$ is a zero divisor. On the other hand, if $\mathfrak{t}(A)$ is locally trivial (i. e. A is locally free), then $A \otimes G \sim A \otimes H$ implies $G \sim H$ by Warfield duality. This says that $[A]x = 0$ implies $x = 0$, so $[A]$ is a regular element of \mathbf{K} . \square

It is shown in [IntCl] that it is possible to partially order the quasi-isomorphism classes of strongly indecomposable modules in such a way that $[G] \leq [H]$ if and only if there is a quasi-epimorphism $G \otimes X \rightarrow H$ for some finite rank torsion free W -module X . In particular, $[G] \leq [H]$ if $[H] \in \text{Support}[G]x$ for some x .

Recall that \mathcal{LF} is the category of locally free modules.

Recall that W -rank $G \neq 0$ if and only if G has a non-trivial projective summand and for $x \in \mathbf{K}$, W -rank $x \neq 0$ if and only if $1 \in \text{Support } x$.

- 6.2. W -rank.** (1) If W -rank $G \neq 0$, then $[G]$ is regular in \mathbf{K} .

- (2) If $x \in \mathbf{K}(\mathcal{LF})$ and W -rank $x \neq 0$, then x is regular in $\mathbf{K}(\mathcal{LF})$.

PROOF: (1) If W -rank $G \neq 0$ then G has a projective summand of strictly positive rank m , so $[G] = m + [G_1]$, with W -rank $G_1 = 0$. Now if $[G]y = 0$, then $my = -[G_1]y$. If we choose $[H_1]$ minimal in $\text{Support } y$ (with respect to the partial ordering defined above), we see that H_1 must be isomorphic to a quasi-summand of $G_1 \otimes H_2$ for some $[H_2] \in \text{Support } y$, where $[H_2] \neq [H_1]$ since the coefficients of $[H_1]$ and $[H_2]$ in y have opposite signs. But then $[H_2] < [H_1]$, contradicting the minimality of $[H_1]$ in $\text{Support } y$.

(2) The reasoning is essentially the same, except that in this case we cannot automatically exclude the possibility $[H_1] = [H_2]$. But for locally free modules, this is still a contradiction, since H_1 cannot be a quasi-summand of $G_1 \otimes H_1$ if W -rank $G_1 = 0$. (A split quasi-epimorphism $G_1 \otimes H_1 \rightarrow H_1$

would lead to a non-trivial map $G_1 \rightarrow (\text{End } H_1)/(\text{nil rad})$, which must split since the codomain is projective.) \square

If D is a domain, then for any G we define $D \otimes [G] = [D \otimes G]$, a quasi-isomorphism class of D -modules that depends only on G . Then by linearity we can define $D \otimes x \in \mathbf{K}(D\text{-modules})$ for any $x \in \mathbf{K}$. (For this to be useful, we should assume, without loss of generality, that D is a dedekind domain.)

6.3. D -rank. (1) If $D\text{-rank } G \neq 0$ for some domain D and $[G]y = 0$ for $y \in \mathbf{K}$, then $D \otimes y = 0$.

(2) If $x, y \in \mathbf{K}(\mathcal{LF})$ and $xy = 0$, and if $D\text{-rank } x \neq 0$ for some domain D , then $D \otimes y = 0$.

(3) If $x, y \in \mathbf{K}(\mathcal{LF})$ and $xy = 0$ and $\text{rank } x \neq 0$, then $D\text{-rank } y = 0$ for all D .

PROOF: (1) If $D\text{-rank } G \neq 0$ then the D -module $D \otimes G$ has a non-trivial projective summand and 6.1 above shows that $[D \otimes G]$ is a regular element in $\mathbf{K}(D\text{-modules})$. Since $[D \otimes G](D \otimes y) = 0$, thus $D \otimes y = 0$.

(2) Analogous.

(3) If $D\text{-rank } y \neq 0$ for some D , then by (2), $D \otimes x = 0$, a contradiction since $\text{rank } x \neq 0$. \square

6.4. Locally Free Completely Decomposable Modules. If $x \neq 0 \in \mathbf{K}(\mathcal{LFCD})$ then x is a regular element in $\mathbf{K}(\mathcal{LF})$.

PROOF: Since $\mathbf{K}(\mathcal{LFCD})$ is a subring of the integral group ring of a torsion free abelian group, it is an integral domain. And $\mathbf{K}(\mathcal{LF})$ is a free (hence a fortiori torsion free) $\mathbf{K}(\mathcal{LFCD})$ -module. \square

6.5. Locally Free Modules. Is $\mathbf{K}(\mathcal{LF})$ an integral domain?

It seems to me that it probably is. An easier question would be: If G is a non-trivial locally free module, then is $[G]$ a regular element in $\mathbf{K}(\mathcal{LF})$? In other words, if G , H_1 , and H_2 are all locally free modules, with G non-trivial, is it possible that $G \otimes H_1$ and $G \otimes H_2$ can be quasi-isomorphic without H_1 and H_2 being quasi-isomorphic? One can see that if $G \otimes H_1 \sim G \otimes H_2$ then, at the least, H_1 and H_2 have to have the same rank, the same inner type and outer type (and in fact, the same typeset) etc. This gives a pretty good clue as to how to construct a counter-example, if one exists. What makes it hard, though, is that tensor products are so hard to compute.

6.6. Generalized Butler Modules. If $x \in \mathbf{K}(\mathcal{QDGB})$, then x is a zero divisor in \mathbf{K} if and only if $D\text{-rank } x = 0$ for some D .

PROOF: (\Leftarrow): The set of $x \in \mathbf{K}(\mathcal{QDGB})$ with D -rank $x = 0$ is P_D , a minimal prime ideal in $\mathbf{K}(\mathcal{QDGB})$, hence consists of zero divisors.

(\Rightarrow): From the structure theorems in [Groth], it can be seen that if D -rank $x \neq 0$ for all domains D , then there exists $y \in \mathbf{K}(\mathcal{QDGB})$ such that $xy = m + n$, where m is a non-zero integer and n is a nilpotent element of \mathbf{K} . Since $m + n$ is not a zero divisor in \mathbf{K} , neither is x . \square

References

- [Groth] E. L. Lady, Grothendieck rings for certain categories of quasi-homomorphisms of torsion free modules over dedekind domains, *J. Algebra* **78** (1982), 273-81.
- [IntCl] E. L. Lady, The integral closure of \mathbb{Z} in the Grothendieck ring for quasi-isomorphism classes of finite rank torsion free modules over dedekind domains, *J. Algebra* **112** (1988), 265-70.
- [Sem] E. L. Lady, A seminar on splitting rings for torsion free modules over dedekind domains, *in* *Abelian Group Theory, Lecture Notes in Mathematics* **1006** (1983), 1-48.
- [WD&ROQ] E. L. Lady, Warfield duality and rank-one quasi-summands of tensor products of finite rank locally free modules over dedekind domains, *J. Algebra* **121** (1989), 129-38.