4. THE t-SOCLE AND t-RADICAL

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Recall Example 1.45. In that example we constructed a rank two module G for which there existed three primes p_1, p_2, p_3 such that the three submodules $G(p_i^\infty G)$ had rank one and were distinct. We were then able to argue that if $\varphi \in \operatorname{End} G$ then φ must leave the three subspaces $Q(p_i^\infty G)$ distinct and that therefore φ must be given by multiplication by a scalar in Q, so that $\operatorname{End} G \subseteq Q$. In particular, we concluded that $\operatorname{End} G$ has no proper itempotents so that G is indecomposable.

There are variations of this example for which the concept $p^\infty G$ is not adequate.

EXAMPLE 4.1. Suppose that $\operatorname{Spec} W$ is infinite and let P_1, P_2, and P_3 be three mutually disjoint infinite sets of primes. Let G be the submodule of $Q \oplus Q$ generated by $W \oplus W$ together with $p_1^{-1}(1,0)$ for all $p_1 \in P_1$, $p_2^{-1}(0,1)$ for all $p_2 \in P_2$, and $p_3^{-1}(1,1)$ for all $p_3 \in P_3$. Then $p^{-\infty}G = 0$ for all primes p. However we can note that if $\mathbf{t}_1 = \mathbf{t}(1,0)$, $\mathbf{t}_2 = \mathbf{t}(0,1)$, and $\mathbf{t}_3 = \mathbf{t}(1,1)$ then these are three mutually incomparable types. We see that for each $i = 1, 2, 3$ the set of elements $g \in G$ with type $\mathbf{t}_i$ forms a pure rank-one submodule of G which by Proposition 2.21 must be invariant under any endomorphism of G. Just as in Example 1.45 we can then argue that any $\varphi \in \operatorname{End} G$ must have three different eigenvectors, hence must be given by multiplication by a scalar, so that $\operatorname{End} G$ is isomorphic to a subring of Q.

In fact, as shown in Example 2.28 we could take any three non-trivial submodules A_1, A_2, A_3 of Q such that the three types $\mathbf{t}_i = \mathbf{t}(A_i)$ are mutually incomparable and such that $A_i \cap A_j$ is the same for all pairs $i \neq j$. We can then construct a module $G \subseteq Q \oplus Q$ by setting $G = A_1(1,0) + A_2(0,1) + A_3(1,1)$. Again, we see that the three rank-one submodules $A_1(1,0)$, $A_2(0,1)$, and $A_3(1,1)$ must be fully invariant, so that $\operatorname{End} G$ is isomorphic to a subring of Q.

The relevant concept here is as follows.

DEFINITION 4.2. Let A be a rank-one module and $\mathbf{t} = \mathbf{t}(A)$. For any torsion free module G we define $G(\mathbf{t}) = \{g \in G \mid \mathbf{t}(g) \geq \mathbf{t}\}$. $G(\mathbf{t})$ is sometimes called the **t-socle** of G. It is a fully invariant pure submodule of G (see Proposition 4.1 and Proposition 4.2 below). In Example 2.28, for instance, as described above, $A_1(1,0) = G(\mathbf{t}_1)$, $A_2(0,1) = G(\mathbf{t}_2)$. and $A_3(1,1) = G(\mathbf{t}_3)$.

PROPOSITION 4.3. *Let* $\mathbf{t} = \mathbf{t}(A)$. *Define* $\sigma\colon A \otimes \operatorname{Hom}(A,G) \to G$ *to be the evaluation map* $a \otimes g \mapsto g(a)$.

(1) σ *is a monomorphism with image* $G(\mathbf{t})$.
(2) $G(\mathbf{t})$ *is a pure submodule of* G.
(3) $G(\mathbf{t})$ *is generated by the set of rank-one submodules of* G *isomorphic to* A.

PROOF: (2) If $g \in G(\mathbf{t})$ and $g' \in G \cap Qg$ then by Proposition 2.* $\mathbf{t}(g') = \mathbf{t}(g) \geq \mathbf{t}$ so $g' \in G(\mathbf{t})$. Thus $G \cap Q(G(\mathbf{t})) \subseteq G(\mathbf{t})$ so $G(\mathbf{t})$ is a pure submodule of G.

(3) If $g \in G$ then $\mathbf{t}(g) = \mathbf{t}(G \cap Qg)$. Thus $\mathbf{t}(g) \geq \mathbf{t}$ if and only if $\mathbf{t}(G \cap Qg) \geq \mathbf{t}$ and if this is so then $G \cap Qg \subseteq G(\mathbf{t})$ because $G \cap Qg$ has rank-one and by (2) $G(\mathbf{t}) \lhd G$ (see Proposition 2.2). Thus every element of $G(\mathbf{t})$ is contained in a pure rank-one submodule B of G with $\mathbf{t}(B) \geq \mathbf{t}$. It thus suffices to see that if B is such a submodule of G then B is generated by rank-one submodules isomorphic to A. At this point we may assume WLOG that A and B are submodules of Q. Then in fact by Proposition 2.11 B is generated by all modules of the form wA for $w \in [B\colon A]$. Since $wA \approx A$, this gives the result.

(1) We may suppose $A \subseteq Q$. As usual, we make the identification $\operatorname{Hom}(A,G) \subseteq \operatorname{Hom}(QA,QG)$. Then σ is the restriction of the evaluation map $\sigma'\colon Q \otimes \operatorname{Hom}(Q,QG) \to QG$. But σ' is known to be an isomorphism. Thus σ is monic.

On the other hand, part (3) shows that $G(\mathbf{t})$ is generated by all submodules of the form $\varphi(A)$ for $\varphi \in \operatorname{Hom}(A,G)$, and this says that σ is surjective. Therefore σ is an isomorphism. $\boxed{\checkmark}$

Analogously to the $\mathbf{t}$-socle we define the $\mathbf{t}$-radical of a torsion free module by setting $G[\mathbf{t}] = \bigcap\{\operatorname{Ker}\varphi \mid \varphi \in \operatorname{Hom}(G,A)\}$, where A is a rank-one module with $\mathbf{t}(A) = \mathbf{t}$. In other words, $G[\mathbf{t}]$ consists of those $g \in G$ which are sent to 0 by every possible homomorphism of G into a rank-one module of type $\mathbf{t}$. We can see from Proposition 2.* that if $0 \neq g \in G$ then $g \in G[\mathbf{t}]$ if and only if whenever H is a pure submodule of G with $g \notin H$ and $\operatorname{rank} G/H = 1$, then $\mathbf{t}(G/H) \not\leq \mathbf{t}$.

PROPOSITION 4.4. (1) *The* $\mathbf{t}$*-socle and* $\mathbf{t}$*-radical are subfunctors of the identity, i.e. if* $\varphi \in \operatorname{Hom}(G,H)$ *then* $\varphi(G(\mathbf{t})) \subseteq H(\mathbf{t})$ *and* $\varphi(G[\mathbf{t}]) \subseteq H[\mathbf{t}]$. *Furthermore if* $\varphi \in \operatorname{QHom}(G,H)$ *then* φ *restricts to quasi-homomorphisms from* $G(\mathbf{t})$ *to* $H(\mathbf{t})$ *and from* $G[\mathbf{t}]$ *to* $H[\mathbf{t}]$.

(2) *If* G *and* H *are quasi-equal then* $G(\mathbf{t})$ *and* $H(\mathbf{t})$ *are quasi-equal, and likewise for* $G[\mathbf{t}]$ *and* $H[\mathbf{t}]$.
(3) *The* $\mathbf{t}$*-socle and* $\mathbf{t}$*-radical respect direct sums, i.e.* $(G_1 \oplus G_2)(\mathbf{t}) = G_1(\mathbf{t}) \oplus G_2(\mathbf{t})$ *and* $(G_1 \oplus G_2)[\mathbf{t}] = G_1[\mathbf{t}] \oplus G_2[\mathbf{t}]$.
(4) *If* $H \lhd G$ *then* $H(\mathbf{t}) = H \cap G(\mathbf{t})$.
(5) *If* $H \subseteq G[\mathbf{t}]$ *then* $G[\mathbf{t}]/H = (G/H)[\mathbf{t}]$.

PROOF: (1) Let $\varphi\colon G \to H$. If $g \in G(\mathbf{t})$ then $\mathbf{t}(g) \geq \mathbf{t}$ and hence by Proposition 2.21 $\mathbf{t}(\varphi(g)) \geq \mathbf{t}$, so that $\varphi(g) \in H(\mathbf{t})$. Now let $g \in G[\mathbf{t}]$ and let A be a rank-one module with $\mathbf{t}(A) = \mathbf{t}$. If $\gamma \in \operatorname{Hom}(H,A)$ then $\gamma\varphi \in \operatorname{Hom}(G,A)$ so $\gamma\varphi(g) = 0$ since $g \in G[\mathbf{t}]$, showing that $\varphi(g) \in H[\mathbf{t}]$.

(2) As shown in (1), the $\mathbf{t}$-socle and $\mathbf{t}$-radical are functors, and clearly are additive functors. (More specifically, we can think of the $\mathbf{t}$-socle, for instance, as a functor $\mathbf{F}$ where $\mathbf{F}(G) = G(\mathbf{t})$ and if $\varphi\colon G \to H$ then $\mathbf{F}(\varphi)\colon G(\mathbf{t}) \to H(\mathbf{t})$ is simply the restriction of $\mathbf{F}$.) Since additive functors respect direct sums (Proposition 0.*) it follows that the $\mathbf{t}$-socle and $\mathbf{t}$-radical respect direct sums.

(3) By Proposition 2.21 if $h \in H$ then $\mathbf{t}_H(h) = \mathbf{t}_G(h)$ so that $h \in H(\mathbf{t}) \iff \mathbf{t}_H(h) \geq \mathbf{t} \iff \mathbf{t}_G(h) \geq \mathbf{t} \iff h \in G(\mathbf{t})$.

(4) Let $K \subseteq G$ be the inverse image of $(G/H)[\mathbf{t}]$ in G and let A be a rank-one module with $\mathbf{t}(A) = \mathbf{t}$. If $H \subseteq G[\mathbf{t}]$ and $\varphi\colon G \to A$ then $\varphi(H) = 0$, so that φ induces $\bar{\varphi}\colon G/H \to A$. Then $\varphi(K) = \bar{\varphi}((G/H)[\mathbf{t}]) = 0$. Since this is true for every $\varphi \in \operatorname{Hom}(G, A)$, $K \subseteq G[\mathbf{t}]$. On the other hand, if $\gamma\colon G \to G/H$ is the quotient map then by (1) $\gamma(G[\mathbf{t}]) \subseteq (G/H)[\mathbf{t}]$ so that $G[\mathbf{t}] \subseteq K$. Thus $K = G[\mathbf{t}]$ and so $(G/H)[\mathbf{t}] = K/H = G[\mathbf{t}]/H$. $\boxed{\checkmark}$

PROPOSITION 4.5. *Let A be a rank-one module and $\mathbf{t} = \mathbf{t}(A)$. Define $\rho\colon G \to \operatorname{Hom}(\operatorname{Hom}(G, A), A)$ by $\rho(g)(\varphi) = \varphi(g)$.*

(1) $\operatorname{Ker}\rho = G[\mathbf{t}]$.
(2) *$G[\mathbf{t}]$ is the intersection of all pure submodules H of G such that $\operatorname{rank} G/H = 1$ and $\mathbf{t}(G/H) \leq \mathbf{t}$.*
(3) $\operatorname{rank} G[\mathbf{t}] = \operatorname{rank} G - \operatorname{rank}\operatorname{Hom}(G, A)$.

PROOF: (1) $g \in G[\mathbf{t}] \iff \varphi(g) = 0$ for all $\varphi \in \operatorname{Hom}(G, A) \iff \rho(g)(\varphi) = 0$ for all $\varphi \in \operatorname{Hom}(G, A) \iff \rho(g) = 0$.

✠ (2) This just rephrases the definition of $G[\mathbf{t}]$. **Explain!**

(3) If $\varphi \neq 0 \in \operatorname{Hom}(G, A)$ then $\operatorname{rank}\operatorname{Ker}\varphi = \operatorname{rank} G - 1$. It follows that in order for $G[\mathbf{t}]$ to be the intersection of $\operatorname{Ker}\varphi$ for various $\varphi \in \operatorname{Hom}(G, A)$ there must be at least $\operatorname{rank} G - \operatorname{rank} G[\mathbf{t}]$ linearly independant φ of this sort. In other words, $\operatorname{rank}\operatorname{Hom}(G, A) \geq \operatorname{rank} G - \operatorname{rank} G[\mathbf{t}]$. On the other hand, since $\varphi(G[\mathbf{t}]) = 0$ for all $\varphi \in G[\mathbf{t}]$, $\operatorname{Hom}(G, A) \approx \operatorname{Hom}(G/G[\mathbf{t}], A)$ so $\operatorname{rank}\operatorname{Hom}(G, A) \leq \operatorname{rank} G/G[\mathbf{t}] = \operatorname{rank} G - \operatorname{rank} G[\mathbf{t}]$. $\boxed{\checkmark}$

Assertions (3) and (4) in Proposition 4.4 need not hold without the hypotheses that $\mathbf{T}(G)$ and $\mathbf{CT}(G)$ are finite respectively. For instance if $\mathbf{t} = \mathbf{IT}(G)$ then $G(\mathbf{s}) \subsetneq G(\mathbf{t}) = G$ for every $\mathbf{s} > \mathbf{t}$. But Example 2.* is an example of a module G with infinite typeset such that $\mathbf{IT}(G) \notin \mathbf{T}(G)$.

EXAMPLES 4.6. (1) Consider the module $G = A_1(1,0) + A_2(0,1) + A_3(1,1)$ in Example 2.28, as described in the beginning of this chapter. $\mathbf{OT}(G)$ consists of $\mathbf{t}_1 \vee \mathbf{t}_2 \vee \mathbf{t}_3$ and $\mathbf{t}_i \vee \mathbf{t}_j$ for $i \neq j = 1, 2, 3$. Then $G[\mathbf{t}_1 \vee \mathbf{t}_2 \vee \mathbf{t}_3] = 0$, $G[\mathbf{t}_1 \vee \mathbf{t}_2] = A_3$, $G[\mathbf{t}_1 \vee \mathbf{t}_3] = A_2$, and $G[\mathbf{t}_2 \vee \mathbf{t}_3] = A_1$.

(2) Let G be the Pontryagin module (Example 1.47). If $\mathbf{t}$ is p-divisible then $G[\mathbf{t}] = 0$, otherwise $G[\mathbf{t}] = G$.

PROOF: (1) If $\mu(G)$ is a rank-one homomorphic image of G let $B_i = \mu(A_i(x,y))$ for $i = 1,2,3$ and x, $y = 0$, 1 chosen appropriately so that $A_i(x,y) \subseteq G$. (Here, as previously, $A_i(x,y)$ denotes the set of elements (ax, ay) for $a \in A_i$.) Then since $G = A_1(1,0) + A_2(0,1) + A_3(1,1)$, $\mu(G) = B_1 + B_2 + B_3$ so that $\mathbf{t}(\mu(G))$ is $\mathbf{t}_1 \vee \mathbf{t}_2$, $\mathbf{t}_1 \vee \mathbf{t}_3$, $\mathbf{t}_2 \vee \mathbf{t}_3$ or $\mathbf{t}_1 \vee \mathbf{t}_2 \vee \mathbf{t}_3$, depending on whether one or none of the B_i are trivial. Now B_1, for instance, is trivial if and only if $A_1(1,0) \subseteq \operatorname{Ker}\mu$. Since $\mathbf{t}_i \vee \mathbf{t}_1 \not\leq \mathbf{t}_2 \vee \mathbf{t}_3$ for $i = 1,2$, it follows that $\mathbf{t}(\mu(G)) \leq \mathbf{t}_2 \vee \mathbf{t}_3 \iff A_1 \subseteq \operatorname{Ker}\mu$ so that $G[\mathbf{t}_2 \vee \mathbf{t}_3] = A_1(1,0)$. And analogously for the other possibilities.

(2) G is p-reduced and $\text{p-rank}\, G = 1 < \operatorname{rank} G$. Thus if $H \lhd G$ then $0 < \text{p-rank}\, H \leq \text{p-rank}\, G = 1$, so that $\text{p-rank}\, H = \text{p-rank}\, G$ and $\text{p-rank}\, G/H = 0$, so that G/H is p-divisible. In particular, if A is a rank-one module which is not p-divisible then $\operatorname{Hom}(G,A) = 0$. Thus $G[\mathbf{t}] = G$ for all types $\mathbf{t}$ which are not p-divisible.

On the other hand, if p' is a prime different from p then $p'\text{-rank}\, G = \operatorname{rank} G$, so that by Proposition 1.25 every rank-one homomorphic image of $G_{p'}$ is a free $W_{p'}$-module. It follows that if G/H is a rank-one homomorphic image of G then $G/H \approx p^{-\infty}$ and so $\mathbf{t}(G/H) \leq \mathbf{t}$ for any p-divisible type $\mathbf{t}$. It then follows from Proposition 4.5 that $G[\mathbf{t}] = 0$.

✠ **FIX THIS!** ☑

PROPOSITION 4.7. (1) *If $\mathbf{s} \leq \mathbf{t}$ then $G(\mathbf{s}) \supseteq G(\mathbf{t})$ and $G[\mathbf{s}] \supseteq G[\mathbf{t}]$.*

(2) $G(\mathbf{s} \vee \mathbf{t}) = G(\mathbf{s})(\mathbf{t}) = G(\mathbf{s}) \cap G(\mathbf{t})$.

(3) $G[\mathbf{s}](\mathbf{t}) = G[\mathbf{s}] \cap G(\mathbf{t})$.

(4) *If $\mathbf{T}(G)$ is finite then $\mathbf{t} \in \mathbf{T}(G)$ if and only if $G(\mathbf{t}) \supsetneq G(\mathbf{s})$ for every $\mathbf{s} > \mathbf{t}$.*

(5) *If $\mathbf{CT}(G)$ is finite then $\mathbf{t} \in \mathbf{CT}(G)$ if and only if $G[\mathbf{t}] \subsetneq G[\mathbf{s}]$ for every $\mathbf{s} < \mathbf{t}$.*

(6) *If $\mathbf{s} \not\leq \mathbf{t}$ then $G(\mathbf{s}) \subseteq G[\mathbf{t}]$.*

PROOF: (1) If $\mathbf{s} \leq \mathbf{t}$ and $g \in G(\mathbf{t})$ then $\mathbf{t}(g) \geq \mathbf{t} \geq \mathbf{s}$ so that $g \in G(\mathbf{s})$.

Now let A and B be rank-one modules such that $\mathbf{t}(A) = \mathbf{s}$ and $\mathbf{t}(B) = \mathbf{t}$. If $\mathbf{s} \leq \mathbf{t}$ we may suppose WLOG that $A \subseteq B$. Then $\operatorname{Hom}(G,A) \subseteq \operatorname{Hom}(G,B)$ so if $g \in G[\mathbf{t}]$ and thus $\gamma(g) = 0$ for all $\gamma \in \operatorname{Hom}(G,B)$ then a fortiori $\gamma(g) = 0$ for all $\gamma \in \operatorname{Hom}(G,A)$, so that $g \in G[\mathbf{s}]$.

(2) By definition of $\mathbf{s} \vee \mathbf{t}$, for $g \in G$, $\mathbf{t}(g) \geq \mathbf{s} \vee \mathbf{t} \iff \mathbf{t}(g) \geq \mathbf{s}$ & $\mathbf{t}(g) \geq \mathbf{t}$. Thus $G(\mathbf{s} \vee \mathbf{t}) = G(\mathbf{s}) \cap G(\mathbf{t})$. Furthermore since $G(\mathbf{s}) \lhd G$, $G(\mathbf{s})(\mathbf{t}) = G(\mathbf{s}) \cap G(\mathbf{t})$ by Proposition 4.4.

(3) Since $G[\mathbf{s}]$ is a pure submodule of G, $G[\mathbf{s}](\mathbf{t}) = G[\mathbf{s}] \cap G(\mathbf{t})$ by Proposition 4.4.

(4) If $\mathbf{t} \in \mathbf{T}(G)$ then there exists $g \in G$ with $\mathbf{t}(g) = \mathbf{t}$, and so $g \in G(\mathbf{t})$ but $g \notin G(\mathbf{s})$ for $\mathbf{s} > \mathbf{t}$. Thus $G(\mathbf{t}) \subsetneq G(\mathbf{s})$. Conversely if $\mathbf{t} \notin \mathbf{T}(G)$ then no element of G has type $\mathbf{t}$, so if $g \in G(\mathbf{t})$ then $\mathbf{t}(g) > \mathbf{t}$, which means that $QG(\mathbf{t}) = \bigcup_{\mathbf{s}>\mathbf{t}} QG(\mathbf{s})$, where the union can be taken over $\mathbf{s}$ with $\mathbf{s} \in \mathbf{T}(G)$ and $\mathbf{s} > \mathbf{t}$. If $\mathbf{T}(G)$ is finite then since a vector space cannot be a finite union of proper subspaces we conclude that $QG(\mathbf{t}) = QG(\mathbf{s})$ for some $\mathbf{s} > \mathbf{t}$. Since $G(\mathbf{t})$ and $G(\mathbf{s})$ are pure submodules of G, we conclude that $G(\mathbf{t}) = G(\mathbf{s})$.

(5) ($\Leftarrow$): If $G[\mathbf{t}] \subsetneq G[\mathbf{s}]$ for every $\mathbf{s} < \mathbf{t}$ then by Proposition 4.5, $\operatorname{rank}\operatorname{Hom}(G,B) < \operatorname{rank}\operatorname{Hom}(G,A)$ so $\operatorname{QHom}(G,B) \subsetneq \operatorname{QHom}(G,A)$ whenever A is a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ and B a rank-one module with $\mathbf{t}(B) = \mathbf{s} < \mathbf{t}$. Now if

$\mathbf{CT}(G)$ is finite, then there are only finitely many B with $\mathbf{t}(B) \in \mathbf{CT}(G)$ and $\mathbf{t}(B) < \mathbf{t}$ **(Explain!)**. Since a vector space over the infinite field Q cannot be a union of a finite number of proper subspaces, thus there exists $\varphi \in \mathrm{Hom}(G, A)$ with $\varphi \notin \mathrm{QHom}(G, B)$ for any B with $\mathbf{t}(B) < \mathbf{t}$. Thus $\mathbf{t}(\varphi(A)) = \mathbf{t}$ so $\mathbf{t} \in \mathbf{CT}(G)$.

($\Rightarrow$): If $\mathbf{t} \in \mathbf{CT}(G)$ and $\mathbf{s} < \mathbf{t}$ and $B \subsetneq A$ are rank-one modules with $\mathbf{t}(A) = \mathbf{t}$, $\mathbf{t}(B) = \mathbf{s}$, then there exists $\varphi \in \mathrm{Hom}(G, A)$ with $\varphi(G) = A$. Thus $\varphi \notin \mathrm{Hom}(G, B)$ so that $\mathrm{Hom}(G, B) \subsetneq \mathrm{Hom}(G, A)$. Thus by Proposition 4.6 $\operatorname{rank} G[\mathbf{t}] < \operatorname{rank} G[\mathbf{s}]$ so that $G[\mathbf{t}] \subsetneq G[\mathbf{s}]$.

(6) If $\mathbf{s} \not\leq \mathbf{t}$ and A is a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ and $\varphi \in \mathrm{Hom}(G, A)$ then $A(\mathbf{s}) = 0$ and so by Proposition 4.4 $\varphi(G(\mathbf{s})) \subseteq A(\mathbf{s}) = 0$. Thus $G(\mathbf{s}) \subseteq G[\mathbf{t}]$. $\boxed{\checkmark}$

DOMINATION. We say that H **dominates** G if $\mathbf{IT}(H) \geq \mathbf{OT}(G)$.

PROPOSITION 4.8. (1) *H dominates G if and only if* $\mathrm{QHom}(G, H) = \mathrm{Hom}(QG, QH)$.

(2) *If H dominates G then G is isomorphic to a submodule of H^r, for any r such that $r(\operatorname{rank} H) \geq \operatorname{rank} G$.*

(3) *Domination is transitive, i.e. if K dominates H and H dominates G, then K dominates G.*

PROOF: (1) Since $\mathrm{Hom}(QG, QH)$ is generated by those linear transformations $QG \to QH$ with one-dimensional image, $\mathrm{QHom}(G, H) = \mathrm{Hom}(QG, QH)$ if and only if $\mathrm{QHom}(G, H)$ contains every $\varphi\colon QG \to QH$ with $\dim \varphi(QG) = 1$, i.e. if and only if for every $\varphi\colon G \to QH$ with $\operatorname{rank} \varphi(G) = 1$ there exists $w \neq 0 \in W$ with $w\varphi(G) \subseteq H$. The latter condition holds if and only if $\mathbf{t}(\varphi(G)) \leq \mathbf{t}(H \cap \varphi(QG))$. So $\mathrm{QHom}(G, H) = \mathrm{Hom}(QG, QH) \iff$ the type of every rank-one homomorphic image of G is less than or equal to the type of every pure rank-one submodule of H, i.e. $\mathbf{s} \leq \mathbf{t}$ for all $\mathbf{s} \in \mathbf{CT}(G)$, $\mathbf{t} \in \mathbf{T}(H)$. But this is equivalent to saying $\mathbf{OT}(G) \leq \mathbf{IT}(H)$. **(Rewrite this!)**

(2) If $r(\operatorname{rank} H) \geq \operatorname{rank} G$ then $\dim QG \leq \dim QH^r$, so QG is isomorphic to a subspace of QH^r. If H dominates G then clearly H^r also dominates G and so if $\varphi\colon QG \to QH^r$ is a monomorphism then $\varphi \in \mathrm{QHom}(G, H^r)$. Then for some $w \neq 0 \in W$, $w\varphi$ is a monomorphism from G into H^r.

(3) Recall from Proposition 2.33 that $\mathbf{IT}(H) \leq \mathbf{OT}(H)$. Thus if K dominates H and H dominates G then $\mathbf{OT}(G) \leq \mathbf{IT}(H) \leq \mathbf{OT}(H) \leq \mathbf{IT}(K)$, so that K dominates G. $\boxed{\checkmark}$

PROPOSITION 4.9. *Let H dominate G and let p be a prime. Then*

(1) *Either H is p-divisible or G_p is a free W_p-module.*

(2) $\text{p-rank}\, \mathrm{Hom}(G, H) = (\text{p-rank}\, G)(\text{p-rank}\, H)$.

(3) $\mathrm{Hom}(G, H)_p = \mathrm{Hom}(G_p, H_p)$.

PROOF: (1) If H is not p-divisible then $\mathbf{IT}(H)$ is not p-divisible. If H dominates G, then $\mathbf{OT}(G) \leq \mathbf{IT}(H)$, so $\mathbf{OT}(G)$ is not p-divisible. Thus by Proposition 2.33 $\text{p-rank}\, G = \operatorname{rank} G$ so that G_p is a free W_p-module by Proposition 1.25.

(3) By Proposition 1.39, $\mathrm{Hom}(G,H)_p \lhd \mathrm{Hom}(G_p,H_p)$. But since H dominates G, $\mathrm{Hom}(QG,QH) = \mathrm{QHom}(G,H) \subseteq \mathrm{QHom}(G_p,H_p) \subseteq \mathrm{Hom}(QG,QH)$, so that $\mathrm{Hom}(G,H)$ is also an essential submodule of $\mathrm{Hom}(G_p,H_p)$. Hence by Proposition 1.17 they are identical.

(2) By Proposition 1.24, $\text{p-rank}\,\mathrm{Hom}(G,H) = \text{p-rank}\,\mathrm{Hom}(G,H)_p$. By (3) $\mathrm{Hom}(G,H)_p = \mathrm{Hom}(G_p,H_p)$. Thus by (1) we need consider only the case where G_p is a free W_p-module or the case where H_p is divisible. In the first case $\mathrm{Hom}(G_p,H_p) \approx H_p^r$, with $r = \mathrm{rank}\,G = \text{p-rank}\,G$, and in the second case by Corollary 1.34 $\mathrm{Hom}(G_p,H_p)$ is also divisible, so that $\text{p-rank}\,\mathrm{Hom}(G,H) = 0$. In either case, the result is clear. ☑

Proposition 4.9 shows that if G and H are p-local modules then H dominates G if and only if either H is divisible or G is a free W_p-module. Thus domination is not an interesting concept for p-local modules.

PROPOSITION 4.10. *Let K dominate G.*

(1) *If $G' \lhd G$ then the restriction map $\mathrm{Hom}(G,K) \to \mathrm{Hom}(G',K)$ is a surjection.*
(2) *If $\varphi\colon K \to K'$ is a surjection, then $\varphi_*\colon \mathrm{Hom}(G,K) \to \mathrm{Hom}(G,K')$ is surjective.*
(3) *If $\varphi\colon K \to G$ is a surjection, then φ splits.*
(4) *If $K \lhd G$ then K is a summand of G.*

PROOF: (1) By Proposition 0.* it suffices to prove that for each prime p the map $\mathrm{Hom}(G,K)_p \to \mathrm{Hom}(G',K)_p$ is surjective. But by Proposition 4.9 we can write this as $\mathrm{Hom}(G_p,K_p) \to \mathrm{Hom}(G'_p,K_p)$. Furthermore by Proposition 4.9 either K_p is divisible, hence injective, in which case clearly this map is a surjection, or G_p is projective, in which case G_p/G'_p is also projective and G'_p is a summand of G_p and once again the map is a surjection.

(2) Analogous. It suffices to prove that for each p the map $\mathrm{Hom}(G_p,K_p) \to \mathrm{Hom}(G_p,K'_p)$ is surjective. This is clear if G_p is free. On the other hand, if K_p is divisible then so is K'_p and the map in question simply becomes $\mathrm{Hom}(QG,QK) \to \mathrm{Hom}(QG,QK')$, which is a surjection by vector space theory.

(3) Applying (2) with $K' = G$ we see that the induced map $\varphi_*\colon \mathrm{Hom}(G,K) \to \mathrm{Hom}(G,G)$ is surjective. In particular, $1_G = \varphi_*(\delta) = \varphi\delta$ for some $\delta \in \mathrm{Hom}(G,K)$, so that φ splits.

(4) Applying (1) with $G' = K$ we see that the restriction map $\mathrm{Hom}(G,K) \to \mathrm{Hom}(K,K)$ is surjective, so that there exists $\gamma \in \mathrm{Hom}(G,K)$ so that the restriction of γ to K is the identity. Thus γ is a projection from G onto K. ☑

It needs to be noted that parts (3) and (4) of Proposition 4.10 are stated in somewhat fraudulent generality. If G is a homomorphic image of H and H dominates G then G dominates itself, since $\mathbf{OT}(G) \le \mathbf{OT}(H) \le \mathbf{IT}(G)$. Likewise if $H \lhd G$ and H dominates G then H dominates itself. But we will see shortly that the only modules which dominate themselves are those which are direct sums of mutually quasi-isomorphic rank-one modules (see Corollary 4.15 below).

PROPOSITION 4.11. *If H dominates G then for any finite rank torsion free module K, $H \otimes K$ dominates G and* $\mathrm{Hom}(G, H \otimes K) \approx \mathrm{Hom}(G, H) \otimes K$.

PROOF: (1) By Proposition 2.35 $\mathbf{IT}(H \otimes K) = \mathbf{IT}(H)\mathbf{IT}(K) \geq \mathbf{IT}(H) \geq \mathbf{OT}(G)$.

(2) By Proposition 1.40 the canonical map $\mathrm{Hom}(G, H) \otimes K \to \mathrm{Hom}(G, H \otimes K)$ is a monomorphism with pure image. But since H dominates G, $\mathrm{rank}\,\mathrm{Hom}(G, H) \otimes K = (\mathrm{rank}\, G)(\mathrm{rank}\, H)(\mathrm{rank}\, K) \geq \mathrm{rank}(\mathrm{Hom}(G, H) \otimes K)$. Therefore this map must also be surjective. ☑

t-SATURATED AND t-BOUNDED MODULES. We say that G is **t-saturated** if $G(\mathbf{t}) = G$ and **t-bounded** if $G[\mathbf{t}] = 0$. We say that G is **t-projective** if it is completely decomposable and homogeneous of type $\mathbf{t}$. In other words, G is $\mathbf{t}$-projective if $G = \bigoplus A_i$, where the A_i are rank-one modules with $\mathbf{t}(A_i) = \mathbf{t}$.

PROPOSITION 4.12. (1) *Pure submodules and homomorphic images of* $\mathbf{t}$*-saturated modules are* $\mathbf{t}$*-saturated. Furthermore if G is an essential submodule of H and G is* $\mathbf{t}$*-saturated then H is* $\mathbf{t}$*-saturated.*

(2) *Submodules and homomorphic images of* $\mathbf{t}$*-bounded modules are* $\mathbf{t}$*-bounded.*

(3) *A module quasi-isomorphic to a* $\mathbf{t}$*-saturated module is* $\mathbf{t}$*-saturated and a module quasi-isomorphic to a* $\mathbf{t}$*-bounded module is* $\mathbf{t}$*-bounded.*

(4) *Let H be a pure submodule of G. Then G/H is* $\mathbf{t}$*-bounded* $\iff H \supseteq G[\mathbf{t}]$.

(5) *If $G_1, \ldots, G_n$ are* $\mathbf{t}$*-saturated submodules of G, then $G_1 + \cdots + G_n$ is* $\mathbf{t}$*-saturated.*

(6) *If $G_1, \ldots, G_n$ are* $\mathbf{t}$*-bounded submodules of G, then $G_1 + \cdots + G_n$ is* $\mathbf{t}$*-bounded.*

✠ PROOF: (1) If G is $\mathbf{t}$-saturated and $\varphi\colon G \to K$ is surjective then by Proposition 4.4 $K(\mathbf{t}) \supseteq \varphi(G(\mathbf{t})) = \varphi(G) = K$, so K is $\mathbf{t}$-saturated. And if $H \lhd G$ then by Proposition 4.4 $H(\mathbf{t}) = H \cap G(\mathbf{t}) = H \cap G = H$, so H is $\mathbf{t}$-saturated. And if G is an essential submodule of H then $G = G(\mathbf{t}) \subseteq H(\mathbf{t})$, so that $H(\mathbf{t})$ is an essential submodule of H. Since $H(\mathbf{t}) \lhd H$, $H = H(\mathbf{t})$, so H is $\mathbf{t}$-saturated.

(2) By Proposition 2.33 if $H \subseteq G$ then $\mathbf{OT}(H) \leq \mathbf{OT}(G)$ and if $H \lhd G$ then $\mathbf{OT}(G/H) \leq \mathbf{OT}(G)$. It then follows from (?next proposition?) that if G is bounded then so are H and G/H.

(3) If G and H are quasi-isomorphic and G is $\mathbf{t}$-saturated then we may assume that G is an essential submodule of H. Therefore by (1) H is $\mathbf{t}$-saturated. Likewise if H is $\mathbf{t}$-bounded then by (5) G is $\mathbf{t}$-bounded.

(4) ($\Leftarrow$): Let A be a rank-one module with $\mathbf{t} = \mathbf{t}(A)$. Since $\mathrm{Hom}(G/G[\mathbf{t}], A) \approx \mathrm{Hom}(G, A)$, by Proposition 4.5 $\mathrm{rank}\,\mathrm{Hom}(G/G[\mathbf{t}], A) = \mathrm{rank}\,\mathrm{Hom}(G, A) = \mathrm{rank}\, G/G[\mathbf{t}] = \mathrm{rank}\,\mathrm{Hom}(G/G[\mathbf{t}], Q) = \mathrm{rank}\,\mathrm{Hom}(G/G[\mathbf{t}], QA)$, so that by Proposition 4.8 A dominates $G/G[\mathbf{t}]$, so $\mathbf{OT}(G/G[\mathbf{t}] \leq \mathbf{IT}(A) = \mathbf{t}$. But if $H \supseteq G[\mathbf{t}]$ then G/H is a homomorphic image of $G/G[\mathbf{t}]$ so by Proposition 2.33, $\mathbf{OT}(G/H) \leq \mathbf{OT}(G/G[\mathbf{t}]) \leq \mathbf{t}$ and so G/H is $\mathbf{t}$-bounded.

($\Rightarrow$): If G/H is $\mathbf{t}$-bounded let $g \in G[\mathbf{t}]$. Let A be a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ and $\bar{\varphi}\colon G/H \to A$. Then $\bar{\varphi}$ induces $\varphi\colon G \to A$ and since $g \in G[\mathbf{t}]$, then $\varphi(g) = 0$. Thus $\bar{\varphi}(g + H) = 0$. Since this is true for every $\bar{\varphi}$ mapping G into A then $g + H \in (G/H)[\mathbf{t}] = 0$, i.e. $g \in H$. Thus $G[\mathbf{t}] \subseteq H$.

(5) & (6) $G_1 + \cdots + G_n$ is a homomorphic image of $G_1 \oplus \cdots \oplus G_n$, which is $\mathbf{t}$-saturated or $\mathbf{t}$-bounded if each G_i is. Thus the result follows from (1) and (2). ☑

PROPOSITION 4.13. (1) *G is $\mathbf{t}$-saturated if and only if $\mathbf{IT}(G) \geq \mathbf{t}$.*

(2) *G is $\mathbf{t}$-bounded if and only if $\mathbf{OT}(G) \leq \mathbf{t}$.*

(3) *If A is a rank-one module with $\mathbf{t} = \mathbf{t}(A)$ then G is $\mathbf{t}$-saturated if and only if G dominates A and G is $\mathbf{t}$-bounded if and only if A dominates G.*

(4) *A module is $\mathbf{t}$-saturated if and only if it is a homomorphic image of a (possibly infinite) direct sum of rank-one modules of type $\mathbf{t}$.*

(5) *A module is $\mathbf{t}$-bounded if and only if it is isomorphic to a submodule of a $\mathbf{t}$-projective module.*

(6) *If $\mathbf{t}$ is not p-divisible and G is $\mathbf{t}$-bounded then $\text{p-rank}\, G = \text{rank}\, G$ and G_p is a free W_p-module.*

(7) *If G is $\mathbf{t}$-bounded and H $\mathbf{t}$-saturated, then H dominates G.*

PROOF: (1) G is $\mathbf{t}$-saturated if and only if $G = G(\mathbf{t})$, i.e. if and only if $\mathbf{t}(g) \geq \mathbf{t}$ for all $g \in G$. This is true if and only if $\mathbf{IT}(G) \geq \mathbf{t}$.

(2) ($\Leftarrow$): Suppose $\mathbf{OT}(G) \leq \mathbf{t}$ and let $g \neq 0 \in G$. Then there exists a map $\varphi\colon G \to Q$ with $\varphi(g) \neq 0$. Since $\mathbf{OT}(G) \leq \mathbf{t}$, $\mathbf{t}(\varphi(G)) \leq \mathbf{t}$. Thus there is a non-trivial map from G into the rank-one module $\varphi(G)$ with $\mathbf{t}(\varphi(G)) \leq \mathbf{t}$, so $g \notin G[\mathbf{t}]$. Thus $G[\mathbf{t}] = 0$, so G is $\mathbf{t}$-bounded.

($\Rightarrow$): Let G be $\mathbf{t}$-bounded and let A be a rank-one module such that $\mathbf{t}(A) = \mathbf{t}$. Then by Proposition 4.5 $\text{rank}\,\text{Hom}(G, A) = \text{rank}\, G/G[\mathbf{t}] = \text{rank}\, G = \text{rank}\,\text{Hom}(G, Q) = \text{rank}\,\text{Hom}(G, QA)$. Thus $\text{QHom}(G, A) = \text{Hom}(G, QA)$ so that by Proposition 4.8, A dominates G, i.e. $\mathbf{OT}(G) \leq \mathbf{IT}(A) = \mathbf{t}(A)$.

(3) This is just a restatement of (1) and (2).

(4) Let G be $\mathbf{t}$-saturated. Then G is a homomorphic image of $\bigoplus\{A \subseteq G \mid \text{rank}\, A = 1 \text{ and } \mathbf{t}(A) = \mathbf{t}\}$. The converse follows from Proposition 4.12 (1).
✠ ????

(5) ($\Rightarrow$): By (3) G is $\mathbf{t}$-bounded if and only if A dominates G, where $\mathbf{t} = \mathbf{t}(A)$. Thus by Proposition 4.8 G is isomorphic to a submodule of A^r, i.e. to a submodule of a $\mathbf{t}$-projective module of rank r.

($\Leftarrow$): It A is a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ then A is $\mathbf{t}$-bounded by (1). It follows that $\mathbf{t}$-projective modules are $\mathbf{t}$-bounded, hence submodules of $\mathbf{t}$-projective modules are $\mathbf{t}$-bounded by Proposition 4.12 (2).

(6) If G is $\mathbf{t}$-bounded and $\mathbf{t}$ is not p-divisible, then $\mathbf{OT}(G)$ is not p-divisible since $\mathbf{OT}(G) \leq \mathbf{t}$. Thus by Proposition 2.33 $\text{p-rank}\, G = \text{rank}\, G$. Then by Proposition 1.25, G_p is a free W_p-module.

(7) If G is $\mathbf{t}$-bounded and H is $\mathbf{t}$-saturated, then $\mathbf{OT}(G) \leq \mathbf{t} \leq \mathbf{IT}(H)$. Hence H dominates G. ☑

The following Proposition will partially justify the use of the term "$\mathbf{t}$-projective."

PROPOSITION 4.14. *Let A be a subring of Q, let $\mathbf{t} = \mathbf{t}(A)$, and let G is any finite rank torsion free module.*

(1) *G is $\mathbf{t}$-saturated if and only if G is isomorphic to an A-module.*
(2) *G is $\mathbf{t}$-projective if and only if G is isomorphic to a projective A-module.*

PROOF: (1) If G is an A-module and $g \neq 0 \in G$ then $Ag \subseteq G \cap Qg$ and since $Ag \approx A$ we see that $\mathbf{t}(A) \leq \mathbf{t}(G \cap Qg) = \mathbf{t}(g)$, so $g \in G(\mathbf{t})$. Thus G is $\mathbf{t}$-saturated. Conversely if G is $\mathbf{t}$-saturated then we will show that G is an A-submodule of QG, i.e that $AG \subseteq G$. If $g \neq 0 \in G$ then $Ag \subseteq QG$ and $Ag \approx A$. Then $\mathbf{t}(G \cap Ag) = \mathbf{t}(G \cap Qg \cap Ag) = \mathbf{t}(g) \wedge \mathbf{t}(A) = \mathbf{t}(A)$ since $\mathbf{t}(g) \geq \mathbf{t}(A)$. Thus by Proposition 2.15, $G \cap Ag$ is a finitely generated A-submodule of Qg, and hence of QG. Thus $Ag \subseteq A(G \cap Ag) \subseteq G \cap Ag \subseteq G$. Since this holds for all $g \in G$, G is an A-submodule of QG.

(2) By Proposition 0.* A is a dedekind domain. (Recall that by convention the word "subring" means a W-subalgebra, i.e. A is a subring of Q containing W.) Thus an A-module is projective if and only if it is a direct sum of A-ideals. On the other hand, G is $\mathbf{t}$-projective if and only if G is a direct sum of rank-one modules of type $\mathbf{t}$. But by Proposition 2.15 a rank-one module has type $\mathbf{t}$ if and only if it is isomorphic to a fractional ideal in A, and hence to a projective A-module. Hence G is $\mathbf{t}$-projective if and only if G is isomorphic to a projective A-module. ☑

EXAMPLES 4.15. (1) It $\mathbf{t} = \mathbf{t}(W)$ then for any G, $G(\mathbf{t}) = G$ and $G[\mathbf{t}]$ is a summand of G and is the smallest pure submodule H of G such that G/H is projective. Every G is $\mathbf{t}$-saturated and G is $\mathbf{t}$-bounded if and only if it is projective.

(2) If $\mathbf{t} = \mathbf{t}(Q)$ then for any G, $G[\mathbf{t}] = 0$ and $G(\mathbf{t}) = \mathbf{d}(G)$. Every G is $\mathbf{t}$-bounded and G is $\mathbf{t}$-saturated if and only if it is divisible.

PROOF: By Proposition 2.32 $\mathbf{IT}(G) \geq \mathbf{t}(W)$ and $\mathbf{OT}(G) \leq \mathbf{t}(Q)$ for every G, and $\mathbf{OT}(G) = \mathbf{t}(W)$ if and only if G is projective and $\mathbf{IT}(G) = \mathbf{t}(Q)$ if and only if G is divisible. Thus since by Proposition 4.12 (6) $G[\mathbf{t}(W)]$ is the maximal pure submodule H of G such that G/H is $\mathbf{t}$-bounded, it follows that it is the maximal pure submodule such that G/H is projective. Since $G/G[\mathbf{t}]$ is projective, $G[\mathbf{t}]$ is a summand.

Likewise $G(\mathbf{t}(Q))$ is the maximal $\mathbf{t}(Q)$-saturated submodule of G, i.e. the maximal divisible submodule of G. ☑

BAER'S LEMMA. What makes everything work is the following theorem, due to Reinhold Baer. It is essentially just a special case of Proposition 4.10, however it seems worthwhile to also show how the traditional proof can be adapted to the dedekind domain case.

THEOREM 4.16. *If $H \lhd G$ and G/H is $\mathbf{t}$-bounded and $G = H + G(\mathbf{t})$, then $G = H \oplus L$ for some $L \subseteq G(\mathbf{t})$.*

FIRST PROOF. $G(\mathbf{t})/(G(\mathbf{t}) \cap H) \approx (G(\mathbf{t}) + H)/H = G/H$. Now since $G(\mathbf{t})$ is $\mathbf{t}$-saturated and G/H is $\mathbf{t}$-bounded, $G(\mathbf{t})$ dominates G/H. Thus by Proposition 4.10

$G(\mathbf{t}) \cap H$ is a summand of $G(\mathbf{t})$, say $G(\mathbf{t}) = (G(\mathbf{t}) \cap H) \oplus L$. Then since $L \subseteq G(\mathbf{t})$, $H \cap L = G(\mathbf{t}) \cap H \cap L = 0$ and $H \oplus L = H + (G(\mathbf{t}) \cap H) + L = H + G(\mathbf{t}) = G$.

A MORE TRADITIONAL PROOF: For this proof, we need to assume that G/H is $\mathbf{t}$-projective. It turns out that this involves no loss of generality, since the result in this case is adequate to prove Corollary 4.14 below, which implies that if G/H is $\mathbf{t}$-bounded then it must be $\mathbf{t}$-projective. Assume then that $G/H = \bigoplus A_i$, where the A_i are rank-one modules of type $\mathbf{t}$. Let $\varphi\colon G \to G/H$ be the quotient map. We need to find ✠ $\sigma\colon G/H \to G$ splitting φ. It suffices to find $\sigma_i\colon A_i \to G$ for each i such that $\varphi\sigma_i = 1_{A_i}$.

Now fix i. Since $G = G(\mathbf{t}) + \operatorname{Ker}\varphi$, there exists a pure rank-one submodule B of $G(\mathbf{t})$ with $0 \neq \varphi(B) \subseteq A_i$. Then $\mathbf{t}(B) \geq \mathbf{t}$ and also since $\operatorname{Hom}(B, A_i) \neq 0$, $\mathbf{t}(B) \leq \mathbf{t}$. So $\mathbf{t}(\varphi(B)) = \mathbf{t}(B) = \mathbf{t} = \mathbf{t}(A_i)$. Thus by Proposition 2.2, $A_i/\varphi(B)$ is cyclic. Let $a \in A_i$ be a pre-image for a generator of $A_i/\varphi(B)$. Since $A \subseteq \varphi(G(\mathbf{t}))$ there exists a pure rank-one submodule C of $G(\mathbf{t})$ with $a \in \varphi(C)$. Thus $\varphi(B) + \varphi(C) = A_i$. The restrictions of φ to B and C are not trivial, hence are monic by Proposition 2.*, so there exist $\sigma_B\colon \varphi(B) \to B$ and $\sigma_C\colon \varphi(C) \to C$ such that $\varphi\sigma_B = 1_B$ and $\varphi\sigma_C = 1_C$. Now by Proposition 2.* $[A_i\colon A_i] = [\varphi(B) + \varphi(C)\colon A_i] = [\varphi(B)\colon A_i] + [\varphi(C)\colon A_i]$, so there exist in particular $u \in [\varphi(B)\colon A_i]$ and $v \in [\varphi(C)\colon A_i]$ such that $u + v = 1$. Now define $\sigma\colon A_i \to G$ by $\sigma_i(x) = \sigma_B(ux) + \sigma_C(vx)$. Note that by construction $ux \in \varphi(B)$ and $vx \in \varphi(C)$, and furthermore $\varphi\sigma_i(x) = \varphi\sigma_B(ux) + \varphi\sigma_C(vx) = ux + vx = x$. I.e. $\varphi\sigma_i = 1_{A_i}$.

Now combining these maps σ_i yields a map $\sigma\colon G/H = \bigoplus A_i \to G$ such that $\varphi\sigma = 1_{G/H}$. Thus φ splits and H is a summand of G. ☑

PROPOSITION 4.17. *A finite rank torsion free module G is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded if and only if G is $\mathbf{t}$-projective.*

PROOF: ($\Leftarrow$): Clear.

($\Rightarrow$): By induction on $\operatorname{rank} G$. Suppose that G is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded. If G has rank-one we see that $\mathbf{t}(G) = \mathbf{t}$ and G is trivially $\mathbf{t}$-projective. If $\operatorname{rank} G > 1$, let $H \lhd G$ be such that $\operatorname{rank} G/H = 1$. Then by Proposition 4.12 G/H is $\mathbf{t}$-bounded and so by Baer's Lemma, $G \approx H \oplus G/H$. By induction, H is $\mathbf{t}$-projective. Therefore G is $\mathbf{t}$-projective. ☑

COROLLARY 4.18. (1) *If G and H are finite rank torsion free modules then G and H mutually dominate each other if and only if there exists a type $\mathbf{t}$ such that G and H are each $\mathbf{t}$-projective.*

(2) *If G and H are $\mathbf{t}$-projective modules and $QG = QH$ then G and H are quasi-equal.*

(3) *A finite rank torsion free module G dominates itself if and only if G is $\mathbf{t}$-projective for some type $\mathbf{t}$.*

PROOF: (1) G and H mutually dominate each other if and only if $\mathbf{IT}(G) \leq \mathbf{OT}(G) \leq \mathbf{IT}(H) \leq \mathbf{OT}(H) \leq \mathbf{IT}(G)$ (c.f. Proposition 2.33),

i.e. if and only if G and H are both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded, where $\mathbf{t} = \mathbf{IT}(G) = \mathbf{IT}(H) = \mathbf{OT}(G) = \mathbf{OT}(H)$. By Proposition 4.17 this is equivalent to G and H both being $\mathbf{t}$-projective.

(2) If G and H are $\mathbf{t}$-projective and $QG = QH$, then G and H dominate each other by (1). Thus 1_{QG} belongs to both $\mathrm{QHom}(G, H)$ and $\mathrm{QHom}(H, G)$. Thus G and H are quasi-equal.

(3) This follows by applying (1) to the case $H = G$. ☑

PROPOSITION 4.19. (1) *Pure submodules and torsion free homomorphic images of* $\mathbf{t}$*-projective modules are* $\mathbf{t}$*-projective.*

(2) *A pure submodule of a* $\mathbf{t}$*-projective module is a direct summand.*

(3) *A module quasi-isomorphic to a* $\mathbf{t}$*-projective module is* $\mathbf{t}$*-projective.*

(4) *A* $\mathbf{t}$*-projective submodule* H *of a* $\mathbf{t}$*-projective module* G *is a quasi-summand of* G.

PROOF: (1) By Proposition 4.12 a pure submodule or homomorphic image of a $\mathbf{t}$-projective module is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded. Thus the result follows from Proposition 4.17.

(2) If G is $\mathbf{t}$-projective and $H \lhd G$, then by Proposition 4.17 G is $\mathbf{t}$-saturated and by Proposition 4.12 G/H is $\mathbf{t}$-bounded. Thus H is a summand of G by Baer's Lemma (Theorem 4.16).

(3) If G is $\mathbf{t}$-projective and $H \sim G$, the H is $\mathbf{t}$-bounded and $\mathbf{t}$-saturated, by Proposition 4.17 H is $\mathbf{t}$-projective.

(4) By (2) H_* is a summand of H and hence by (1) is $\mathbf{t}$-projective. Since H and H_* have the same rank, it is thus clear that they are quasi-isomorphic. Thus by Proposition 3.9 the inclusion map $H \hookrightarrow H_*$ is a quasi-isomorphism. Hence since H_* is a summand of G, H is a quasi-summand. ☑

The following proposition is extremely useful for avoiding calculations when constructing examples:

PROPOSITION 4.20. *If* $H \lhd G$ *and* $G = H + K$ *where* K *is* $\mathbf{t}$*-projective for some type* $\mathbf{t}$*, then there exists a pure submodule* K' *of* K *such that* $G = H \oplus K'$.

PROOF: We can apply Baer's Lemma. In fact, since $K \subseteq G(\mathbf{t})$, clearly $G = H + G(\mathbf{t})$ and G/H is $\mathbf{t}$-projective since it is a homomorphic image of K. Thus by Baer's Lemma H is a summand of G. Now we need to see that the complementary summand can be chosen to be a submodule of K. One can verify this rather tediously by retracing the proof of Baer's Lemma. A faster way is to note that by Baer's Lemma $K = (H \cap K) \oplus K'$ for some $K' \subseteq K$. As in the proof of Theorem 4.16 one then sees that $G = H \oplus K'$. ☑

EXAMPLE 4.21. Consider again Example 1.49. In this example H is generated by $B \oplus C$ together with $(b+c)/w$ where B and C are rank-one modules with incomparable types, and $b \in B$, $c \in C$ are such that $b \notin wB$, $c \notin wC$. We let A be a module isomorphic to B and let $a \in A$ correspond to $b \in B$ under some isomorphism. Now let B_2 be the pure rank-one submodule of $A \oplus H$ generated by $a + 2b$, and let H_2 be the purification of $B_2 + C$. Then there is an isomorphism θ from H_2 to the submodule of $QB \oplus QC$ generated by $B \oplus 2C$ together with $(b+2c)/w$, given by $\theta(a+2b) = b$ and $\theta(2c) = c$. We have seen that under suitable hypotheses on A and B, H and H_2 are not isomorphic. But there exists a rank-one submodule A_2 of $A \oplus H$ such that $A_2 \oplus H_2 = A \oplus H$, and if W is a principal ideal domain then $A_2 \approx A$.

PROOF: Let $G = A \oplus H$. Clearly $G = H_1 + (A \oplus B)$. Since $A \oplus B$ is $\mathbf{t}(A)$-projective, by Proposition 4.20 $A \oplus H = A_2 \oplus H_2$ for some rank-one submodule A_2 of $A \oplus B$. By Proposition 4.19 A_2 is $\mathbf{t}$-projective, i.e. $\mathbf{t}(A_2) = \mathbf{t}(A)$. Thus if W is a principal ideal domain then $A_2 \approx A$ by Corollary 2.10. ☑

In contrast to most of the theorems in this chapter and the next, the dual theorem to Baer's Lemma is not valid. The dual theorem would assert that if H is a $\mathbf{t}$-projective pure submodule of G such that $H \cap G[\mathbf{t}] = 0$ then H is a direct summand. We have the following simple counter-example:

EXAMPLE 4.22. Consider again Example 1.49. In this example H is generated by $B \oplus C$ together with $(b+c)/w$ where B and C are rank-one modules with incomparable types, and $b \in B$, $c \in C$ are such that $b \notin wB$, $c \notin wC$. Now let $\mathbf{t} = \mathbf{t}(B)$. Then $H[\mathbf{t}] = C$ and $B \cap H[\mathbf{t}] = 0$. But B is not a direct summand of H.

PROOF: If $\varphi \in \mathrm{Hom}(H, B)$ then $\varphi(C) = 0$ since $\mathbf{t}(C)$ and $\mathbf{t}$ are incomparable. Thus $C \subseteq H[\mathbf{t}]$. On the other hand, since $H/C \approx B$, $H[\mathbf{t}] \subseteq C$ by Proposition 4.12. We have seen earlier that H is indecomposable, so B is not a summand of H. ☑

However we do have the following two propositions, which are slightly weaker.

PROPOSITION 4.23. *Let G be $\mathbf{t}$-bounded and H a pure submodule of G. Let K be a $\mathbf{t}$-saturated module. Then every homomorphism $\varphi\colon H \to K$ extends to a homomorphism $G \to K$.*

PROOF: If G is $\mathbf{t}$-bounded and K $\mathbf{t}$-saturated then K dominates G. Hence this is a special case of Proposition 4.10. ☑

PROPOSITION 4.24. (1) *If H is a $\mathbf{t}$-saturated submodule of G and $H \cap G[\mathbf{t}] = 0$ then H is $\mathbf{t}$-projective and G is quasi-equal to $H \oplus L$ for some submodule L with $G[\mathbf{t}] \subseteq L$ and $H \oplus G[\mathbf{t}]$ is a quasi-pure submodule of G.*

(2) *If in addition $H + G[\mathbf{t}]$ is a pure submodule of G then $G = H \oplus L$ for some L with $G[\mathbf{t}] \subseteq L$.*

(3) *If G is $\mathbf{t}$-bounded and H is a pure $\mathbf{t}$-saturated submodule of G, then H is a $\mathbf{t}$-projective direct summand of G.*

PROOF: (3) If G is $\mathbf{t}$-bounded and H $\mathbf{t}$-saturated, then Proposition 4.23 applied to 1_H shows that H is a summand of G. Since H is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded, it is $\mathbf{t}$-projective by Proposition 4.17.

(2) If $H + G[\mathbf{t}] = H \oplus G[\mathbf{t}]$ is a pure submodule of G then $(H \oplus G[\mathbf{t}])/G[\mathbf{t}]$ is a pure $\mathbf{t}$-saturated submodule of the $\mathbf{t}$-bounded module $G/G[\mathbf{t}]$, and hence is a summand of $G/G[\mathbf{t}]$ by (3). Thus there is a projection $G/G[\mathbf{t}] \to (H \oplus G[\mathbf{t}])/G[\mathbf{t}]$ which leads to a sequence of maps

$$G \to G/G[\mathbf{t}] \to (H \oplus G[\mathbf{t}])/G[\mathbf{t}] \xrightarrow{\approx} H.$$

Let θ be the composition of these maps. Then θ restricts to the identity on H, showing that $G = H \oplus \operatorname{Ker}\theta$. Note that $G[\mathbf{t}] \subseteq \operatorname{Ker}\theta$.

(1) $(H \oplus G[\mathbf{t}])/G[\mathbf{t}]$ is $\mathbf{t}$-bounded since it is a submodule of $G/G[\mathbf{t}]$ and is $\mathbf{t}$-saturated since it is isomorphic to H. And $(H \oplus G[\mathbf{t}])_*/G[\mathbf{t}]$ is also $\mathbf{t}$-bounded and $\mathbf{t}$-saturated. Hence these are both $\mathbf{t}$-projective and hence quasi-isomorphic, since they have the same rank. Thus by Proposition 3.9 they are quasi-equal, so $(H \oplus G[\mathbf{t}])/G[\mathbf{t}]$ is a quasi-pure submodule of $G/G[\mathbf{t}]$. Thus by Proposition 3.14 $H \oplus G[\mathbf{t}]$ is a quasi-pure submodule of G. Now the reasoning in (2) applies, except that in this case we can only conclude that θ is a quasi-projection of G "quasi-onto" H. Thus G is quasi-equal to $H \oplus L$ for some L with $G[\mathbf{t}] \subseteq L$. ☑

PROPOSITION 4.25. *If $\mathbf{t} = \mathbf{OT}(G)$ then $G(\mathbf{t})$ is $\mathbf{t}$-projective and is a summand of G.*

PROOF: Since $\mathbf{t} = \mathbf{OT}(G)$, G is $\mathbf{t}$-bounded. Since $G(\mathbf{t}) \lhd G$, by Proposition 4.12 $G(\mathbf{t})$ is $\mathbf{t}$-bounded. Since it is $\mathbf{t}$-saturated, it is $\mathbf{t}$-projective by Proposition 4.17 and is a summand of G by Proposition 4.24. ☑

PROPOSITION 4.26. *Let G be a finite rank torsion free module.*

(1) *$G(\mathbf{t}) + G[\mathbf{t}] = G_{\mathbf{t}} \oplus G[\mathbf{t}]$, where $G_{\mathbf{t}}$ is a $\mathbf{t}$-projective module.*
(2) *$G(\mathbf{t}) + G[\mathbf{t}]$ is a quasi-pure submodule of G.*
(3) *$G_{\mathbf{t}}$ is a maximal $\mathbf{t}$-projective quasi-summand of G.*
(4) *$(G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$ is a $\mathbf{t}$-projective direct summand of $G/G[\mathbf{t}]$.*
(5) *$G_{\mathbf{t}} \sim (G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$.*

PROOF: (1) $(G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$ is $\mathbf{t}$-bounded since it is a submodule of the $\mathbf{t}$-bounded module $G/G[\mathbf{t}]$. Hence by Theorem 4.16 $G[\mathbf{t}]$ is a summand of $G(\mathbf{t}) + G[\mathbf{t}]$ and the complementary summand is $\mathbf{t}$-projective.

(2) By Proposition 4.24 $G_{\mathbf{t}} \oplus G[\mathbf{t}] = G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-pure in G.

(3) Since $G_{\mathbf{t}}$ is $\mathbf{t}$-saturated and $G_{\mathbf{t}} \cap G[\mathbf{t}] = 0$, by Proposition 4.24 $G_{\mathbf{t}}$ is a quasi-summand of G. Clearly it is a maximal $\mathbf{t}$-projective quasi-summand, since if H is a $\mathbf{t}$-projective quasi-summand of G then $H \subseteq G(\mathbf{t})$ and if $\pi\colon G \to H$ is a quasi-projection then $\pi(G[\mathbf{t}]) \subseteq H[\mathbf{t}] = 0$, so that $H \cap G[\mathbf{t}] = 0$.

(4) $(G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$ is a $\mathbf{t}$-saturated submodule of the $\mathbf{t}$-bounded module $G/G[\mathbf{t}]$. Therefore it is a $\mathbf{t}$-projective direct summand of $G/G[\mathbf{t}]$ by Proposition 4.24.

(5) $G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-equal to $(G(\mathbf{t}) + G[\mathbf{t}])_*$ by (2), so that $G_{\mathbf{t}} \approx (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}] \sim (G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$. ☑

COMPLETELY DECOMPOSABLE MODULES. Recall that a module is called **completely decomposable** if it is a direct sum of rank-one modules. We can now rephrase this by saying that a module is completely decomposable if it is a direct sum of $\mathbf{t}$-projective modules for various $\mathbf{t}$.

By Jónsson's Theorem (Theorem 3.24) a maximal completely decomposable quasi-summand of a module G is determined uniquely up to quasi-isomorphism. It is somewhat surprising, although quite elementary, that the same is true of a maximal completely decomposable actual summand of G.

PROPOSITION 4.27. (1) *A maximal* $\mathbf{t}$*-projective summand of a finite rank torsion free module* G *is unique up to quasi-isomorphism.*

(2) *A maximal completely decomposable summand of* G *is unique up to quasi-isomorphism.*

PROOF: (1) Let $G = C \oplus H = D \oplus K$, where C and D are $\mathbf{t}$-projective and H and K have no non-trivial $\mathbf{t}$-projective summand. Then $H \cap D \lhd D$ so by Proposition 4.19 $H \cap D$ is a summand of D, and hence a summand of G, and hence a summand of H. But by assumption H has no non-trivial $\mathbf{t}$-projective summand. Thus $H \cap D = 0$, so the projection of G onto C restricts to a monomorphism from D into C. Likewise there is a monomorphism from C into D, so that by Proposition 3.1 C and D are quasi-isomorphic.

(2) Let $G = H \oplus \bigoplus C_{\mathbf{t}}$ where each $C_{\mathbf{t}}$ is $\mathbf{t}$-projective and H has no non-trivial completely decomposable summand. Then in particular for each $\mathbf{t}$, H has no $\mathbf{t}$-projective summand so by (1) each $C_{\mathbf{t}}$ is unique up to quasi-isomorphism. Thus $\bigoplus C_{\mathbf{t}}$ is unique up to quasi-isomorphism. ☑

✠ Note that if W is a principal ideal domain then ...

In light of Proposition 4.27 it is very plausible to infer think that if C is a maximal completely decomposable summand of G and D a maximal completely decomposable summand of H, then $C \oplus D$ is a maximal completely decomposable summand of $G \oplus H$. However, this need not be the case. In fact, as the following example shows, it is possible for G and H to both have no non-trivial completely decomposable summand at all and yet for $G \oplus H$ to have a very large one.

EXAMPLE 4.28. Let $A_1, \ldots, A_n$ be rank-one modules with mutually incomparable types, and for each i let B_i be a rank-one module with $B_i \approx A_i$. Suppose further that there exist distinct primes p and q such that none of the A_i is p-divisible or q-divisible, and for each i choose $a_i \in A_i$ with $a_i \notin pA_i$ and $b_i \in B_i$ with $b_i \notin qB_i$. Let G be the submodule of $QA_1 \oplus \cdots \oplus QA_n$ generated by $A_1 \oplus \cdots \oplus A_n$ together with $p^{-1}(a_1 + \cdots + a_n)$ and let H be the submodule of $QB_1 \oplus \cdots \oplus QB_n$ generated by $B_1 \oplus \cdots \oplus B_n$ together with $q^{-1}(b_1 + \cdots + b_n)$. Then G and H are indecomposable, and in particular have no non-trivial completely decomposable summands, but $G \oplus H$ has a completely decomposable summand of rank n.

PROOF: ****

DEFINITION 4.29. We define $G^*(\mathbf{t})$ as the submodule generated by $\{g \in G \mid \mathbf{t}(g) > \mathbf{t}\}$ and $G^*[\mathbf{t}]$ as $\bigcap_{\mathbf{s}<\mathbf{t}} G[\mathbf{s}]$. (In other words, $g \in G^*[\mathbf{t}]$ if and only if $\mu(g) = 0$ whenever $\mu\colon G \to A$ where A is a rank-one module with $\mathbf{t}(A) < \mathbf{t}$.)

PROPOSITION 4.30. (1) $G^*(\mathbf{t}) \subseteq G(\mathbf{t}) \cap G[\mathbf{t}]$.
(2) $G(\mathbf{t}) + G[\mathbf{t}] \subseteq G^*[\mathbf{t}]$

PROOF: (1) Clearly $G^*(\mathbf{t}) \subseteq G(\mathbf{t})$. And for all g with $\mathbf{t}(g) > \mathbf{t}$, if $\varphi\colon G \to A$ with $\mathbf{t}(A) = \mathbf{t}$, then $\varphi(g) = 0$ by Proposition 2.21, so that $g \in G[\mathbf{t}]$. Thus $G^*(\mathbf{t}) \subseteq G[\mathbf{t}]$.

(2) Since by Proposition 4.12 $\mathbf{s} < \mathbf{t} \Rightarrow G[\mathbf{s}] \supseteq G[\mathbf{t}]$, it follows that $G[\mathbf{t}] \subseteq G^*[\mathbf{t}]$. And if $g \in G(\mathbf{t})$ then $g \in G[\mathbf{s}]$ for all $\mathbf{s} < \mathbf{t}$ since $\mathbf{t}(g) \geq \mathbf{t} > \mathbf{s}$ and so if $\varphi\colon G \to A$ with $\mathbf{t}(A) = \mathbf{s}$ then $\varphi(g) = 0$. Thus $g \in G^*[\mathbf{t}]$, and so $G(\mathbf{t}) \subseteq G^*[\mathbf{t}]$. ☑

PROPOSITION 4.31. *Let G be completely decomposable and $G = \bigoplus G_{\mathbf{t}}$, where for each $\mathbf{t}$, $G_{\mathbf{t}}$ is $\mathbf{t}$-projective. If $\mathbf{t}$ is any type then*

(1) $G(\mathbf{t}) = \bigoplus_{\mathbf{s}\geq\mathbf{t}} G_{\mathbf{s}}$.
(2) $G^*(\mathbf{t}) = \bigoplus_{\mathbf{s}>\mathbf{t}} G_{\mathbf{s}}$.
(3) $G[\mathbf{t}] = \bigoplus_{\mathbf{s}\not\leq\mathbf{t}} G_{\mathbf{s}}$.
(4) $G^*[\mathbf{t}] = \bigoplus_{\mathbf{s}\not<\mathbf{t}} G_{\mathbf{s}}$.

PROOF: Since $G_{\mathbf{s}}$ is $\mathbf{t}$-projective it is $\mathbf{t}$-saturated and $\mathbf{t}$-bounded.

(1) $G_{\mathbf{s}}(\mathbf{t}) = G_{\mathbf{s}}$ if $\mathbf{s} \geq \mathbf{t}$ and $G_{\mathbf{s}}(\mathbf{t}) = 0$ if $\mathbf{s} \not\geq \mathbf{t}$. Thus by Proposition 4.4 $G[\mathbf{t}] = \bigoplus G_{\mathbf{s}}[\mathbf{t}] = \bigoplus_{\mathbf{s}\geq\mathbf{t}} G_{\mathbf{s}}$.

(2) $G^*_{\mathbf{s}}(\mathbf{t}) = G_{\mathbf{s}}$ if $\mathbf{s} > \mathbf{t}$ and $G^*_{\mathbf{s}}(\mathbf{t}) = 0$ otherwise. Thus $G^*(\mathbf{t}) = \bigoplus G^*_{\mathbf{s}}(\mathbf{t}) = \bigoplus_{\mathbf{s}>\mathbf{t}} G_{\mathbf{s}}$.

(3) $G_{\mathbf{s}}[\mathbf{t}] = 0$ if $\mathbf{s} \leq \mathbf{t}$ and $G_{\mathbf{s}}[\mathbf{t}] = G_{\mathbf{s}}$ otherwise. Thus $G[\mathbf{t}] = \bigoplus G_{\mathbf{s}}[\mathbf{t}] = \bigoplus_{\mathbf{s}\not\leq\mathbf{t}} G_{\mathbf{s}}$.

(4) $G^*_{\mathbf{s}}[\mathbf{t}] = 0$ if and only if $G_{\mathbf{s}}[\mathbf{t}'] = 0$ for some $\mathbf{t}' < \mathbf{t}$ and this is true if and only if $\mathbf{s} < \mathbf{t}$. Otherwise $G^*_{\mathbf{s}}[\mathbf{t}] = G_{\mathbf{s}}$. Thus $G^*[\mathbf{t}] = \bigoplus G^*_{\mathbf{s}}[\mathbf{t}] = \bigoplus_{\mathbf{s}\not<\mathbf{t}} G_{\mathbf{s}}$. (WHY?) ☑

If $g \in G$ then to say that $g \in G^*[\mathbf{t}]$ is to say that $\varphi(g) = 0$ for every $\varphi\colon G \to Q$ such that $\mathbf{t}(\varphi(g)) < \mathbf{t}$. Clearly this is true if either $g \in G[\mathbf{t}]$ or $\mathbf{t}(g) = \mathbf{t}$. We have already taken note of this above as the formula $G(\mathbf{t}) + G[\mathbf{t}] \subseteq G^*[\mathbf{t}]$ in Proposition 4.30. It seems naively plausible that $G[\mathbf{t}]$ and $G(\mathbf{t})$ would account for all of $G^*[\mathbf{t}]$, i.e. that $G^*[\mathbf{t}] = G(\mathbf{t}) + G[\mathbf{t}]$. Surprisingly enough, though, this plausible equality is valid only if G is completely decomposable.

PROPOSITION 4.32. (1) *A finite rank torsion free module G is completely decomposable if and only if if and only if $G^*[\mathbf{t}] = G(\mathbf{t}) + G[\mathbf{t}]$ for all $\mathbf{t} \in \mathbf{CT}(G)$.*

(2) *G is quasi-isomorphic to a completely decomposable module if and only if $G^*[\mathbf{t}] = (G(\mathbf{t}) + G[\mathbf{t}])_*$ for all $\mathbf{t} \in \mathbf{CT}(G)$.*

PROOF: By Proposition 4.26 $G(\mathbf{t}) + G[\mathbf{t}]$ is a quasi-pure submodule of G, so $G^*[\mathbf{t}] = (G(\mathbf{t}) + G[\mathbf{t}])_*$ if and only if $G^*[\mathbf{t}]$ is quasi-equal to $G(\mathbf{t}) + G[\mathbf{t}]$. For every type $\mathbf{t}$, define $\mathbf{F_t}(G) = G^*[\mathbf{t}]/(G(\mathbf{t}) + G[\mathbf{t}])$. (Note that $\mathbf{F_t}(G)$ need not be torsion free.) Then $\mathbf{F_t}$ is a functor in the obvious way, and respects direct sums since it is additive. The assertion then is that G is completely decomposable $\iff$ $\mathbf{F_t}(G) = 0$ for all $\mathbf{t} \in \mathbf{CT}(G)$ and G is quasi-isomorphic to a completely decomposable module if and only if $\mathbf{F_t}(G)$ has finite length for all $\mathbf{t}$.

($\Rightarrow$): Since $\mathbf{F_t}$ respects direct sums, to see that $\mathbf{F_t}(G) = 0$ for all types $\mathbf{t}$ when G is completely decomposable it suffices to prove that $\mathbf{F_t}(G) = 0$ for modules G with $\operatorname{rank} G = 1$. Now if $\operatorname{rank} G = 1$ and $\mathbf{t} \leq \mathbf{t}(G)$ then $G(\mathbf{t}) = G^*[\mathbf{t}] = G$, so $G = G(\mathbf{t}) + G[\mathbf{t}] = G^*[\mathbf{t}]$ and $\mathbf{F_t}(G) = 0$. On the other hand, if $\mathbf{t} > \mathbf{t}(G)$ then $G^*[\mathbf{t}] = 0$ and again $\mathbf{F_t}(G) = 0$. Finally, if $\operatorname{rank} G = 1$ and $\mathbf{t}$ and $\mathbf{t}(G)$ are incomparable, then $G[\mathbf{t}] = G^*[\mathbf{t}] = G$ so that $G = G[\mathbf{t}] + G(\mathbf{t}) = G^*[\mathbf{t}]$ and $\mathbf{F_t}(G) = 0$ in this case as well.

Now if G is quasi-equal to a completely decomposable module G' then $G^*[\mathbf{t}]$ is quasi-equal to $G'^*[\mathbf{t}]$ and $G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-equal to $G'(\mathbf{t}) + G'[\mathbf{t}]$. Thus if $G^*[\mathbf{t}] = G(\mathbf{t}) + G[\mathbf{t}]$ then $G'^*[\mathbf{t}]$ is quasi-equal to $G'(\mathbf{t}) + G'[\mathbf{t}]$. **(???)**

($\Leftarrow$): Now suppose $\mathbf{F_t}(G)$ has finite length for all $\mathbf{t} \in \mathbf{CT}(G)$. We will prove by induction on rank that G is quasi-isomorphic to a completely decomposable module, and that if in fact $\mathbf{F_t}(G) = 0$ for all $\mathbf{t}$ then G is completely decomposable. If $\operatorname{rank} G = 1$ or, more generally, if $\mathbf{CT}(G) = \{\mathbf{t}\}$ for some $\mathbf{t}$, then $G[\mathbf{t}] = 0$ and $G^*[\mathbf{t}] = G$, so that $\mathbf{F_t}(G) = 0$ implies that $G = G(\mathbf{t})$, so G is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded, hence is $\mathbf{t}$-projective and a fortiori completely decomposable. Now in general note that by Proposition 2.26 $\mathbf{CT}(G)$ must contain minimal elements. If $\mathbf{CT}(G)$ contains more than one element, let $\mathbf{s}$ be minimal in $\mathbf{CT}(G)$. Then $G[\mathbf{s}'] = 0$ for all $\mathbf{s}' < \mathbf{s}$, so $G^*[\mathbf{s}] = G$. The hypothesis applied for $\mathbf{t} = \mathbf{s}$ together with Proposition 4.26 yields that $G = G^*[\mathbf{s}]$ is quasi-equal to $G(\mathbf{s}) + G[\mathbf{s}] = G_\mathbf{s} \oplus G[\mathbf{s}]$ for some $\mathbf{s}$-projective module $G_\mathbf{s}$. Then for any $\mathbf{t}$ the fact that $\mathbf{F_t}$ is an additive functor yields $\mathbf{F_t}(G) = \mathbf{F_t}(G_\mathbf{s}) \oplus \mathbf{F_t}(G[\mathbf{s}])$. Then by induction on rank we conclude that $G[\mathbf{s}]$ is quasi-equal to a completely decomposable module, so that $G_\mathbf{s} \oplus G[\mathbf{s}]$ is quasi-equal to q completely decomposable module, and thus G is quasi-equal to a completely decomposable module and is completely decomposable if $\mathbf{F_t}(G) = 0$ for all $\mathbf{t}$. $\boxed{\checkmark}$

In Proposition 4.19 we saw that a module quasi-isomorphic to a $\mathbf{t}$-projective module is also $\mathbf{t}$-projective. It is not in general true, however, that a module quasi-isomorphic to a completely decomposable module is completely decomposable. For instance in Example 1.49 the module H formed from the completely decomposable module $B \oplus C$ by adjoining an element $(b + c)/w$ is quasi-isomorphic to $B \oplus C$ but is not itself completely decomposable: in fact, it is indecomposable and has rank two. H was obtained by filling in glue in the space between the two incomparable modules B and C. If we had tried to add glue on top of B or C, by adjoining to $B \oplus C$ an element b/w, for instance, with $b \in B$, the construction would not have worked. It would also not have worked if if $\mathbf{t}(B) = \mathbf{t}(C)$ since in that case by Proposition 4.19 H would be $\mathbf{t}$-projective. In fact, if $\mathbf{t}(B) = \mathbf{t}(C)$ and $H = (B \oplus C) + W(b + c)/w$, then let B' be the rank-one pure

submodule of H containing $(b+c)/w$. Then $B' \sim B \sim C$ and $H = B' \oplus C$. We will also see in Chapter 5 that H is necessarily completely decomposable if $\mathbf{t}(B) \leq \mathbf{t}(C)$ (Proposition 5.*).

Modules quasi-isomorphic to completely decomposable modules are called *almost completely decomposable* and will be studied in Chapter 5.

The idea that one can test for complete decomposability of a module G by seeing whether a family of functors $\mathbf{F_t}$ vanish on G is quite fascinating. As an immediate consequence, we get the first part of the following classical theorem.

THEOREM 4.33. *(1) A direct summand of a completely decomposable module is completely decomposable.*

(2) If G is completely decomposable, and $G = \bigoplus G_{\mathbf{t}}$ where the $G_{\mathbf{t}}$ are $\mathbf{t}$-projective modules, then the $G_{\mathbf{t}}$ are uniquely determined up to isomorphism. In fact, $G_{\mathbf{t}} \approx (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}] \approx G(\mathbf{t})/G^*(\mathbf{t})$.

PROOF: (1) For any type $\mathbf{t}$, define the functor $\mathbf{F_t}$ as in the proof of the preceding Proposition. Then G is completely decomposable if and only if $\mathbf{F_t}(G) = 0$ for all $\mathbf{t} \in \mathbf{CT}(G)$. But since $\mathbf{F_t}$ is an additive functor, $\mathbf{F_t}(H \oplus K) = \mathbf{F_t}(H) \oplus \mathbf{F_t}(K)$. Hence if $H \oplus K$ is completely decomposable, then so are H and K.

(2) Let $G = \bigoplus G_{\mathbf{t}}$ where each $G_{\mathbf{t}}$ is $\mathbf{t}$-projective. By Proposition 4.31 $G(\mathbf{t}) = \bigoplus_{\mathbf{s} \geq \mathbf{t}} G_{\mathbf{s}}$ and $G[\mathbf{t}] = \bigoplus_{\mathbf{s} \not\leq t} G_{\mathbf{s}} \supseteq \bigoplus_{\mathbf{s} > \mathbf{t}} G_{\mathbf{s}}$, so that $G(\mathbf{t}) + G[\mathbf{t}] = G_{\mathbf{t}} \oplus G[\mathbf{t}]$. Thus $G_{\mathbf{t}}$ is the same $G_{\mathbf{t}}$ as occurred in Proposition 4.32 and $G_{\mathbf{t}} \approx (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$. Furthermore since $G^*(\mathbf{t}) = \bigoplus_{\mathbf{s} > \mathbf{t}} G_{\mathbf{s}}$, $G_{\mathbf{t}} \approx G(\mathbf{t})/G^*(\mathbf{t})$. ☑

COROLLARY 4.34. *If W is a principal ideal domain and G and H are quasi-isomorphic completely decomposable modules, then G and H are in fact isomorphic.*

PROOF: If $G = \bigoplus G_{\mathbf{t}} \sim H = \bigoplus H_{\mathbf{t}}$, where for each $\mathbf{t}$ the $G_{\mathbf{t}}$ and $H_{\mathbf{t}}$ are $\mathbf{t}$-projective, then by Theorem 3.24 $G_{\mathbf{t}} \sim H_{\mathbf{t}}$ for each $\mathbf{t}$. Thus it suffices to suppose that G and H are $\mathbf{t}$-projective for some type $\mathbf{t}$. By Corollary 2.10 if W is a principal ideal domain then the rank-one summands of G are isomorphic to the rank-one summands of H. Thus $G \approx H$. ☑

WARFIELD ISOMORPHISM THEOREMS. In the results that follow it is worth remembering that by Proposition 2.2, if A is a subring of W then $A = S^{-1}W$ for some multiplicative set S and that for any G, $A \otimes G$ is naturally isomorphic to $S^{-1}G$.

PROPOSITION 4.35. *Let A have rank one and let $A_0 = \operatorname{End} A$. Recall that $\mathbf{t}(A_0) = [\mathbf{t}(A) : \mathbf{t}(A)]$. Let G and H be finite rank torsion free modules.*

(1) $G(\mathbf{t}) \approx A_0 \otimes G(\mathbf{t})$.
(2) $G(\mathbf{t}) \approx A \otimes \operatorname{Hom}(A, G)$.
(3) $\operatorname{Hom}(A,\ A \otimes G) \approx A_0 \otimes G$.
(4) $\operatorname{Hom}(A \otimes G,\ A \otimes H) \approx \operatorname{Hom}(G, A_0 \otimes H) \approx \operatorname{Hom}(A_0 \otimes G,\ A_0 \otimes H)$.

PROOF: (2) Already proved in Proposition 4.3.

(3) Since $\mathbf{t}(A \otimes G) \geq \mathbf{t} = \mathbf{OT}(A)$, $A \otimes G$ dominates A. Thus by Proposition 4.11 $\mathrm{Hom}(A, A \otimes G) \approx \mathrm{Hom}(A, A) \otimes G = A_0 \otimes G$.

(4) By (3) and the adjointness between Hom and the tensor product, $\mathrm{Hom}(A \otimes G,\, A \otimes H) \approx \mathrm{Hom}(G,\, \mathrm{Hom}(A, A \otimes H)) \approx \mathrm{Hom}(G, A_0 \otimes H)$ $\approx \mathrm{Hom}(A_0 \otimes G,\, A_0 \otimes H)$ where the last isomorphism holds by Proposition 0.*, since A_0 is a subring of Q. ☑

PROPOSITION 4.36. *If A is a subring of Q and $\mathbf{t} = \mathbf{t}(A)$ then*

(1) $G(\mathbf{t}) \approx \mathrm{Hom}(A, G)$.
(2) *For any H,* $\mathrm{Hom}(G, H)(\mathbf{t}) = \mathrm{Hom}(G, H(\mathbf{t}))$.
(3) *For any G,* $\mathrm{Hom}(G, A)$ *is a $\mathbf{t}$-projective module.*

PROOF: (1) Since A is a subring of Q then $\mathrm{Hom}(A, G)$ is an A module. Furthermore $A = S^{-1}W$ for some multiplicative set S and so by Proposition 4.7 and Proposition 0.*, $G(\mathbf{t}) \approx A \otimes \mathrm{Hom}(A, G) = A \otimes_A \mathrm{Hom}(A, G) \approx \mathrm{Hom}(A, G)$.

(2) By (1), $\mathrm{Hom}(G, H)(\mathbf{t}) \approx \mathrm{Hom}(A,\, \mathrm{Hom}(G, H)) \approx \mathrm{Hom}(A \otimes G,\, H) \approx$ $\mathrm{Hom}(G,\, \mathrm{Hom}(A, H)) \approx \mathrm{Hom}(G, H(\mathbf{t}))$.

(3) **** ☑

COROLLARY 4.37. *If R is any dedekind domain over W (not necessarily with finite rank) then for any finite rank torsion free W-module G,* $\mathrm{Hom}(G, R)$ *is a projective R-module.*

PROOF: By Proposition 0.*, $\mathrm{Hom}(G, R) \approx \mathrm{Hom}_R(R \otimes G, R)$. But $R \otimes G$ is a finite rank torsion free R-module. Therefore we may as well replace the ground ring by R, in other words no generality is lost in supposing $R = W$ so that the result reduces to proposition 4.36. ☑

The following corollary is essentially just an elaboration of what was proved in Proposition 4.14. We take the opportunity, though, to give a different proof.

COROLLARY 4.38. *Let A be a subring of Q and $\mathbf{t} = \mathbf{t}(A)$. Then for any G the following conditions are equivalent:*

(1) *G is $\mathbf{t}$-saturated.*
(2) *G is an A-submodule of QG.*
(3) $A \otimes G \approx G$.
(4) $\mathrm{Hom}(A, G) \approx G$.

PROOF: (1) $\Rightarrow$ (4): By Corollary 4.37.

(4) $\Rightarrow$ (3): As noted in the proof of Corollary 4.37, $A \otimes \mathrm{Hom}(A, G) \approx \mathrm{Hom}(A, G)$.

(3) $\Rightarrow$ (2): Clear since $A \otimes G$ is an A-submodule of $QA \otimes G$.

(2) $\Rightarrow$ (1): If G is an A-module then by Proposition 0.* $\mathrm{Hom}(A, G) \approx \mathrm{Hom}_A(A, G) \approx$ G. Thus by Corollary 4.37 $G(\mathbf{t}) \approx G$. Thus $\operatorname{rank} G(\mathbf{t}) = \operatorname{rank} G$. Since $G(\mathbf{t})$ is a pure submodule of G, we see that $G(\mathbf{t}) = G$, i.e. G is $\mathbf{t}$-saturated. ☑

COROLLARY 4.39. *Let A be a rank-one module and let G and H be $\mathbf{t}(A)$-saturated modules. Then*

(1) *The map* $\mathrm{Hom}(G,H) \to \mathrm{Hom}(A\otimes G,\, A\otimes H)$ *given by* $\varphi \mapsto 1_A \otimes \varphi$ *is an isomorphism.*
(2) *The map* $\mathrm{Hom}(G,H) \to \mathrm{Hom}(\mathrm{Hom}(A,G),\, \mathrm{Hom}(A,H))$ *given by* $\varphi \mapsto \varphi_*$ *is an isomorphism.*

PROOF: (1) If $A_0 = [A\colon A]$ then by Corollary 4.38 and Proposition 4.35 $\mathrm{Hom}(G,H) \approx \mathrm{Hom}(A_0\otimes G,\, A_0\otimes H) \approx \mathrm{Hom}(A\otimes G,\, A\otimes H)$.

(2) Since by Proposition 4.35 the natural maps $\sigma_G\colon A\otimes \mathrm{Hom}(A,G) \to A$ and $\sigma_H\colon A\otimes \mathrm{Hom}(A,H) \to H$ are isomorphisms, it follows from (1) that

$$\mathrm{Hom}(\mathrm{Hom}(A,G),\, \mathrm{Hom}(A,H)) \approx \mathrm{Hom}(A\otimes \mathrm{Hom}(A,G),\, A\otimes \mathrm{Hom}(A,H)) \approx \mathrm{Hom}(G,H).$$

Let η denote the composition of these isomorphisms and let $\theta\colon \mathrm{Hom}(G,H) \to \mathrm{Hom}(\mathrm{Hom}(A,G),\, \mathrm{Hom}(A,H))$ be given by $\theta(\varphi) = \varphi_*$.

Now let $\varphi \in \mathrm{Hom}(G,H)$ and $g \in G$. Then $\eta\theta(\varphi)(g) = \eta(\varphi_*)(g) = \sigma_H(\varphi_*\sigma_G^{-1}(g)) = \sigma_H\sigma_H^{-1}(\varphi(g)) = \varphi(g)$ by the naturality of σ. Thus $\eta\theta(\varphi) = \varphi$ so that $\eta\theta = 1$.

On the other hand, let $\psi \in \mathrm{Hom}(\mathrm{Hom}(A,G),\, \mathrm{Hom}(A,H))$ and $\mu \in \mathrm{Hom}(A,G)$. Then

$$\begin{aligned}
\theta\eta(\psi)(\mu) &= \theta(\sigma_H(1\otimes\psi)\sigma_G^{-1})(\mu)\\
&= (\sigma_H(1\otimes\psi)\sigma_G^{-1})_*(\mu)\\
&= \sigma_H(1\otimes\psi)\sigma_G^{-1}\mu\\
&= (1\otimes\psi)\sigma_G\sigma_G^{-1}\mu\\
&= \psi(\mu)
\end{aligned}$$

so that $\theta\eta(\psi) = \psi$. Thus η is an isomorphism with inverse θ. ☑

COROLLARY 4.40. *If* $[\mathbf{t}(A)\colon \mathbf{t}(A)] \le \mathbf{IT}(G)$ *then* $\mathrm{Hom}(A,\, A\otimes G) \approx G$ *and* $\mathrm{Hom}(A\otimes G,\, A\otimes H) \approx \mathrm{Hom}(G,H)$.

PROOF: If $A_0 = \mathrm{End}\, A$ then by hypothesis G and H are $\mathbf{t}(A_0)$-saturated. Thus by Proposition 4.* $A_0\otimes G \approx G$ and $A_0\otimes H \approx H$, so the result follows from Proposition 4.35. ☑

COROLLARY 4.41. *Let A be a subring of Q and $\mathbf{t} \ge \mathbf{t}(A)$. Then for any G,*

(1) $A\otimes G[\mathbf{t}] = (A\otimes G)[\mathbf{t}]$.
(2) $A\otimes G[\mathbf{t}]$ *is a summand of* $A\otimes G$.
(3) $A\otimes (G/G[\mathbf{t}])$ *is* $\mathbf{t}$*-projective.*

PROOF: (1) We can identify G with the set of elements $1 \otimes g \in A \otimes G$. Thus if $x \in G[\mathbf{t}]$ then $1 \otimes x \in (W \otimes G)[\mathbf{t}]$ and if $\varphi\colon A \otimes G \to A$ then since by Proposition 0.* φ is A-linear, $\varphi(a \otimes x) = a\varphi(1 \otimes x) = 0$. Thus $A \otimes G[\mathbf{t}] \subseteq (A \otimes G)[\mathbf{t}]$ without any assumptions on $\mathbf{t}$. Therefore by Proposition 4.12 it suffices to prove that $(A \otimes G)/(A \otimes G[\mathbf{t}])$ is $\mathbf{t}$-bounded. In fact, $(A \otimes G)/(A \otimes G[\mathbf{t}]) \approx A \otimes G/G[\mathbf{t}]$ and $\mathbf{OT}(A \otimes G/G[\mathbf{t}]) = \mathbf{OT}(A)\mathbf{OT}(G/G[\mathbf{t}]) = \mathbf{t}(A)\,\mathbf{OT}(G/G[\mathbf{t}]) \leq (\mathbf{t}(A))^2 = \mathbf{t}(A) \leq \mathbf{t}$ since $\mathbf{t}(A)$ is idempotent by Proposition2.*.

(3) By (1), $A \otimes (G/G[\mathbf{t}]) \approx (A \otimes G)/(A \otimes G[\mathbf{t}])$, and this is $\mathbf{t}$-bounded. By Corollary 4.40 $A \otimes (G/G[\mathbf{t}])$ is also $\mathbf{t}$-saturated. Thus by Proposition 4.17 it is $\mathbf{t}$-projective.

(2) By Corollary 4.40 $A \otimes G$ is $\mathbf{t}$-saturated. Since $A \otimes (G/G[\mathbf{t}])$ is $\mathbf{t}$-projective by (3), $A \otimes G[\mathbf{t}]$ is a summand of $A \otimes G$ by Baer's Lemma (Theorem 4.16). ☑

PROPOSITION 4.42. (1) *If K is a $\mathbf{t}$-bounded module, then for any G, $\mathrm{Hom}(G, K)$ is $\mathbf{t}$-bounded.*

(2) *If $\mathbf{t}$ is idempotent and K is $\mathbf{t}$-saturated, then $\mathrm{Hom}(G, K)$ is $\mathbf{t}$-saturated.*

PROOF: (1) Let K be $\mathbf{t}$-bounded. Let $\delta\colon G \to \mathrm{Hom}(\mathrm{Hom}(G, K), K)$ be defined by $\delta(g)(\varphi) = \varphi(g)$. Then by Proposition 4.4, for any $g \in G$, $\delta(g)$ maps $\mathrm{Hom}(G, K)[\mathbf{t}]$ into $K[\mathbf{t}] = 0$. Thus if $\varphi \in \mathrm{Hom}(G, K)[\mathbf{t}]$ and $g \in G$ then $\varphi(g) = \delta(g)(\varphi) = 0$. Since this is true for all $g \in G$, thus $\varphi = 0$. Hence $\mathrm{Hom}(G, K)[\mathbf{t}] = 0$ so $\mathrm{Hom}(G, K)$ is $\mathbf{t}$-bounded.

(2) If $\mathbf{t}$ is idempotent then by Proposition 2.15 $\mathbf{t} = \mathbf{t}(A)$ for some subring A of Q. By Proposition 4.14 if K is saturated then K is an A-module. But then $\mathrm{Hom}(G, K)$ is also an A-module, hence is also $\mathbf{t}$-saturated. ☑

PROPOSITION 4.43. *Let A be a rank-one module and $\mathbf{t} = \mathbf{t}(A)$. Then for any G, $\mathbf{IT}(\mathrm{Hom}(A, G)) = [\mathbf{IT}(G(\mathbf{t}))\colon \mathbf{t}]$.*

PROOF: First note that $\mathrm{Hom}(A, G) = \mathrm{Hom}(A, G(\mathbf{t}))$. If B is a pure rank-one submodule of $G(\mathbf{t})$, then by Proposition 1.37 $\mathrm{Hom}(A, B)$ is a pure rank-one submodule of $\mathrm{Hom}(A, G)$. Furthermore these are all the pure rank-one submodules of $\mathrm{Hom}(A, G)$ since if $\varphi \in \mathrm{Hom}(A, G)$ then $\varphi \in \mathrm{Hom}(A, \varphi(A)_*)$. But by Proposition 2.*, $\mathbf{t}(\mathrm{Hom}(A, B)) = [\mathbf{t}(B)\colon \mathbf{t}]$. Taking the greatest lower bound over all $B \lhd G(\mathbf{t})$ yields $\mathbf{IT}(\mathrm{Hom}(A, G)) = [\mathbf{IT}(G(\mathbf{t}))\colon \mathbf{t}]$. ☑

COROLLARY 4.44. *For any W-module G, there exists a rank-one module A with locally trivial type and a module G_0 with $\mathbf{IT}(G_0) = [\mathbf{IT}(G)\colon \mathbf{IT}(G)]$ (so that, in particular, $\mathbf{IT}(G_0)$ is idempotent) such that $G \approx A \otimes G_0$.*

PROOF: Let $\mathbf{t}' = \mathbf{IT}(G)$, let $\mathbf{t}_0 = [\mathbf{t}'\colon \mathbf{t}']$, and let A' be a rank-one module with $\mathbf{t}' = \mathbf{t}(A')$. Let $G_0 = \mathrm{Hom}(A', G)$. By Proposition 4.43, $\mathbf{IT}(G_0) = [\mathbf{IT}(G(\mathbf{t}))\colon \mathbf{t}'] = [\mathbf{IT}(G)\colon \mathbf{t}'] = [\mathbf{t}'\colon \mathbf{t}'] = \mathbf{t}_0$. By Proposition 4.35, $G = G(\mathbf{t}') \approx A' \otimes \mathrm{Hom}(A', G) = A' \otimes G_0$. Now by Proposition 2.19 $\mathbf{t}'$ can be written $\mathbf{t}' = \mathbf{t}[\mathbf{t}'\colon \mathbf{t}'] = \mathbf{t}\mathbf{t}_0$, where $\mathbf{t}$ is locally trivial. Thus if A is a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ and if $A_0 = [A'\colon A']$ (so that $\mathbf{t}(A_0) = \mathbf{IT}(G_0)$) then $G \approx A' \otimes G_0 \approx A \otimes A_0 \otimes G_0 \approx A \otimes G_0$, since $A \otimes A_0 \approx A_0 A = A$. (WHY?) ☑

We define the $\mathbf{t}$-rank of G as $\operatorname{rank}\operatorname{Hom}(G,A)$ where A is a rank-one module with $\mathbf{t}(A)=\mathbf{t}$.

PROPOSITION 4.45. *Let A be a rank-one module with type $\mathbf{t}$.*

(1) $\mathbf{t}\text{-}\operatorname{rank} G = \operatorname{rank} G/G[\mathbf{t}]$.
(2) $\operatorname{Hom}(G,A)$ *is* $\mathbf{t}$*-bounded.*
(3) *If* $\mathbf{t}$ *is idempotent, then* $\operatorname{Hom}(G,A)$ *is* $\mathbf{t}$*-projective.*
(4) *If* $\mathbf{t}$ *is idempotent then* $\mathbf{t}\text{-}\operatorname{rank} G$ *is the rank of a maximal* $\mathbf{t}$*-projective quasi-direct summand of* $A\otimes G$.
(5) *If* $\mathbf{t}$ *is idempotent then* $\mathbf{t}\text{-}\operatorname{rank}(G\otimes H) = (\mathbf{t}\text{-}\operatorname{rank} G)(\mathbf{t}\text{-}\operatorname{rank} H)$.

PROOF: (1) This was proved in Proposition 4.5.

(2) This is a special case of Proposition 4.42.

(3) If $\mathbf{t}$ is idempotent, then by Proposition 4.42 $\operatorname{Hom}(G,A)$ is $\mathbf{t}$-saturated. Since it is $\mathbf{t}$-bounded by (2), by Proposition 4.17 it is $\mathbf{t}$-projective.

(4) If $\mathbf{t}$ is idempotent then $\operatorname{Hom}(G,A) \approx \operatorname{Hom}_A(A\otimes G, A)$ so that $\mathbf{t}\text{-}\operatorname{rank} G = \mathbf{t}\text{-}\operatorname{rank}(A\otimes G)$ and in we lose no generality in thinking of A as the ground ring. In other words, WLOG we may suppose $A = W$. But then $\mathbf{t}\text{-}\operatorname{rank} G = \operatorname{rank}\operatorname{Hom}(G,W) = \operatorname{rank} G/G[\mathbf{t}]$. But by Proposition 4.26 $G/G[\mathbf{t}] = G(\mathbf{t})/G[\mathbf{t}]$ is isomorphic to a maximal $\mathbf{t}$-projective summand of G.

(5) $\operatorname{Hom}(G\otimes H,\, A) \approx \operatorname{Hom}(G, \operatorname{Hom}(H,A))$. Now $\operatorname{Hom}(H,A)$ is an A-module, hence is $\mathbf{t}$-projective by (2), i.e. is a direct sum of s rank-one modules with type $\mathbf{t}$, where $s = \mathbf{t}\text{-}\operatorname{rank} H$. Thus $\mathbf{t}\text{-}\operatorname{rank}(G\otimes H) = \operatorname{rank}\operatorname{Hom}(G\otimes H,\, A) = \operatorname{rank}\operatorname{Hom}(G,\, \operatorname{Hom}(H,A)) = (\mathbf{t}\text{-}\operatorname{rank} H)(\operatorname{rank}\operatorname{Hom}(G,A)) = (\mathbf{t}\text{-}\operatorname{rank} G)(\mathbf{t}\text{-}\operatorname{rank} H)$. ☑

PROPOSITION 4.46. *Let* $\mathbf{t} = \mathbf{IT}(G)$. *Then*

(1) $G = G_{\mathbf{t}} \oplus G[\mathbf{t}]$, *for some* $\mathbf{t}$*-projective module* $G_{\mathbf{t}}$.
(2) $\mathbf{t}\text{-}\operatorname{rank} G$ *is the rank of a maximal* $\mathbf{t}$*-projective quasi-summand of* G. *And, in fact,* G *has a* $\mathbf{t}$*-projective actual summand of this rank.*

PROOF: (1) Since $\mathbf{t} = \mathbf{IT}(G)$, $G = G(\mathbf{t}) = G(\mathbf{t}) + G[\mathbf{t}]$. Thus by Proposition 4.26 $G = G_{\mathbf{t}} \oplus G[\mathbf{t}]$, where $G_{\mathbf{t}}$ is $\mathbf{t}$-projective.

(2) By Proposition 4.26 $G_{\mathbf{t}}$ is a maximal $\mathbf{t}$-projective quasi-summand of G. Since by Proposition 4.45 $\mathbf{t}\text{-}\operatorname{rank} G = \operatorname{rank} G/G[\mathbf{t}] = \operatorname{rank} G_{\mathbf{t}}$, the result follows. And by (1), G actually has a $\mathbf{t}$-projective summand of this rank. ☑