Recall Example 1.45. In that example we constructed a rank two module $G$ for which there existed three primes $p_1$, $p_2$, $p_3$ such that the three submodules $G(p_i^\infty G)$ had rank one and were distinct. We were then able to argue that if $\varphi \in \text{End} G$ then $\varphi$ must leave the three subspaces $Q(p_i^\infty G)$ distinct and that therefore $\varphi$ must be given by multiplication by a scalar in $Q$, so that $\text{End} G \subseteq Q$. In particular, we concluded that $\text{End} G$ has no proper idempotents so that $G$ is indecomposable.

There are variations of this example for which the concept $p_1 G$ is not adequate.

**Example 4.1.** Suppose that $\text{Spec} \ W$ is infinite and let $P_1$, $P_2$, and $P_3$ be three mutually disjoint infinite sets of primes. Let $G$ be the submodule of $Q \oplus Q$ generated by $W \oplus W$ together with $p_1^{-1}(1,0)$ for all $p_1 \in P_1$, $p_2^{-1}(0,1)$ for all $p_2 \in P_2$, and $p_3^{-1}(1,1)$ for all $p_3 \in P_3$. Then $p^{-\infty}G = 0$ for all primes $p$. However we can note that if $t_1 = t(1,0)$, $t_2 = t(0,1)$, and $t_3 = t(1,1)$ then these are three mutually incomparable types. We see that for each $i = 1, 2, 3$ the set of elements $g \in G$ with type $t_i$ forms a pure rank-one submodule of $G$ which by Proposition 2.21 must be invariant under any endomorphism of $G$. Just as in Example 1.45 we can then argue that any $\varphi \in \text{End} G$ must have three different eigenvectors, hence must be given by multiplication by a scalar, so that $\text{End} G$ is isomorphic to a subring of $Q$.

In fact, as shown in Example 2.28 we could take any three non-trivial submodules $A_1$, $A_2$, $A_3$ of $Q$ such that the three types $t_i = t(A_i)$ are mutually incomparable and such that $A_i \cap A_j$ is the same for all pairs $i \neq j$. We can then construct a module $G \subseteq Q \oplus Q$ by setting $G = A_1(1,0) + A_2(0,1) + A_3(1,1)$. Again, we see that the three rank-one submodules $A_1(1,0)$, $A_2(0,1)$, and $A_3(1,1)$ must be fully invariant, so that $\text{End} G$ is isomorphic to a subring of $Q$.

The relevant concept here is as follows.

**Definition 4.2.** Let $A$ be a rank-one module and $t = t(A)$. For any torsion free module $G$ we define $G(t) = \{g \in G \mid t(g) \geq t\}$. $G(t)$ is sometimes called the $t$-**socle** of $G$. It is a fully invariant pure submodule of $G$ (see Proposition 4.1 and Proposition 4.2 below). In Example 2.28, for instance, as described above, $A_1(1,0) = G(t_1)$, $A_2(0,1) = G(t_2)$, and $A_3(1,1) = G(t_3)$. 

Proposition 2.2). Thus every element of $G$ is generated by rank-one submodules isomorphic to $B$.

Proof: (2) If $g \in G(t)$ and $g' \in G \cap Qg$ then by Proposition 2.* $t(g') = t(g) \geq t$ so $g' \in G(t)$. Thus $G \cap Q(G(t)) \subseteq G(t)$ so $G(t)$ is a pure submodule of $G$.

(3) If $g \in G$ then $t(g) = t(G \cap Qg)$. Thus $t(g) \geq t$ if and only if $t(G \cap Qg) \geq t$ and if this is so then $G \cap Qg \subseteq G(t)$ because $G \cap Qg$ has rank-one and by (2) $G(t) \lhd G$ (see Proposition 2.2). Thus every element of $G(t)$ is contained in a pure rank-one submodule $B$ of $G$ with $t(B) \geq t$. It thus suffices to see that if $B$ is such a submodule of $G$ then $B$ is generated by rank-one submodules isomorphic to $A$. At this point we may assume wlog that $A$ and $B$ are submodules of $Q$. Then in fact by Proposition 2.11 $B$ is generated by all modules of the form $wA$ for $w \in [B : A]$. Since $wA \approx A$, this gives the result.

(1) We may suppose $A \subseteq Q$. As usual, we make the identification

$\text{Hom}(A, G) \subseteq \text{Hom}(QA, QG)$. Then $\sigma$ is the restriction of the evaluation map $\sigma': Q \otimes \text{Hom}(Q,G) \to QG$. But $\sigma'$ is known to be an isomorphism. Thus $\sigma$ is monic.

On the other hand, part (3) shows that $G(t)$ is generated by all submodules of the form $\varphi(A)$ for $\varphi \in \text{Hom}(A, G)$, and this says that $\sigma$ is surjective. Therefore $\sigma$ is an isomorphism. \(\Box\)

Analogously to the $t$-socle we define the $t$-radical of a torsion free module by setting $G[t] = \bigcap \{ \ker \varphi \mid \varphi \in \text{Hom}(G, A) \}$, where $A$ is a rank-one module with $t(A) = t$.

In other words, $G[t]$ consists of those $g \in G$ which are sent to 0 by every possible homomorphism of $G$ into a rank-one module of type $t$. We can see from Proposition 2.* that if $0 \neq g \in G$ then $g \in G[t]$ if and only if whenever $H$ is a pure submodule of $G$ with $g \notin H$ and rank $G/H = 1$, then $t(G/H) \leq t$.

Proposition 4.4. (1) The $t$-socle and $t$-radical are subfunctors of the identity, i.e.

if $\varphi \in \text{Hom}(G, H)$ then $\varphi(G(t)) \subseteq H(t)$ and $\varphi(G[t]) \subseteq H[t]$. Furthermore if $\varphi \in \text{QHom}(G, H)$ then $\varphi$ restricts to quasi-homomorphisms from $G(t)$ to $H(t)$ and from $G[t]$ to $H[t]$.

(2) If $G$ and $H$ are quasi-equal then $G(t)$ and $H(t)$ are quasi-equal, and likewise for $G[t]$ and $H[t]$.

(3) The $t$-socle and $t$-radical respect direct sums, i.e. $(G_1 \oplus G_2)(t) = G_1(t) \oplus G_2(t)$ and $(G_1 \oplus G_2)[t] = G_1[t] \oplus G_2[t]$.

(4) If $H \lhd G$ then $H(t) = H \cap G(t)$.

(5) If $H \subseteq G[t]$ then $G[t]/H = (G/H)[t]$.

Proof: (1) Let $\varphi: G \to H$. If $g \in G(t)$ then $t(g) \geq t$ and hence by Proposition 2.21 $t(\varphi(g)) \geq t$, so that $\varphi(g) \in H(t)$. Now let $g \in G[t]$ and let $A$ be a rank-one module with $t(A) = t$. If $\gamma \in \text{Hom}(H, A)$ then $\gamma \varphi \in \text{Hom}(G, A)$ so $\gamma \varphi(g) = 0$ since $g \in G[t]$, showing that $\varphi(g) \in H[t]$.
(2) As shown in (1), the \( t \)-socle and \( t \)-radical are functors, and clearly are additive functors. (More specifically, we can think of the \( t \)-socle, for instance, as a functor \( F \) where \( F(G) = G(t) \) and if \( \varphi : G \to H \) then \( F(\varphi) : G(t) \to H(t) \) is simply the restriction of \( F \).) Since additive functors respect direct sums (Proposition 0.*) it follows that the \( t \)-socle and \( t \)-radical respect direct sums.

(3) By Proposition 2.21 if \( h \in H \) then \( t_H(h) = t_G(h) \) so that \( h \in H(t) \iff t_H(h) \geq t \iff t_G(h) \geq t \iff h \in G(t) \).

(4) Let \( K \subseteq G \) be the inverse image of \( (G/H)[t] \) in \( G \) and let \( A \) be a rank-one module with \( t(A) = t \). If \( H \subseteq G[t] \) and \( \varphi : G \to A \) then \( \varphi(H) = 0 \), so that \( \varphi \) induces \( \varphi : G/H \to A \). Then \( \varphi(K) = \varphi((G/H)[t]) = 0 \). Since this is true for every \( \varphi \in \text{Hom}(G, A) \), \( K \subseteq G[t] \). On the other hand, if \( \gamma : G \to G/H \) is the quotient map then by (1) \( \gamma(G[t]) \subseteq (G/H)[t] \) so that \( G[t] \subseteq K \). Thus \( K = G[t] \) and so \( (G/H)[t] = K/H = G[t]/H \).

**Proposition 4.5.** Let \( A \) be a rank-one module and \( t = t(A) \). Define \( \rho : G \to \text{Hom}(\text{Hom}(G, A), A) \) by \( \rho(g)(\varphi) = \varphi(g) \).

(1) \( \ker \rho = G[t] \).

(2) \( G[t] \) is the intersection of all pure submodules \( H \) of \( G \) such that \( \text{rank} G/H = 1 \) and \( t(G/H) \leq t \).

(3) \( \text{rank} G[t] = \text{rank} G - \text{rank} \text{Hom}(G, A) \).

**Proof:** (1) \( g \in G[t] \iff \varphi(g) = 0 \) for all \( \varphi \in \text{Hom}(G, A) \iff \rho(g)(\varphi) = 0 \) for all \( \varphi \in \text{Hom}(G, A) \iff \rho(g) = 0 \).

(2) This just rephrases the definition of \( G[t] \). **Explain!**

(3) If \( \varphi \neq 0 \in \text{Hom}(G, A) \) then \( \text{rank} \ker \varphi = \text{rank} G - 1 \). It follows that in order for \( G[t] \) to be the intersection of \( \ker \varphi \) for various \( \varphi \in \text{Hom}(G, A) \) there must be at least \( \text{rank} G - \text{rank} G[t] \) linearly independent \( \varphi \) of this sort. In other words, \( \text{rank} \text{Hom}(G, A) \geq \text{rank} G - \text{rank} G[t] \). On the other hand, since \( \varphi(G[t]) = 0 \) for all \( \varphi \in G[t] \), \( \text{Hom}(G, A) \approx \text{Hom}(G/G[t], A) \) so \( \text{rank} \text{Hom}(G, A) \leq \text{rank} G/G[t] = \text{rank} G - \text{rank} G[t] \).

Assertions (3) and (4) in Proposition 4.4 need not hold without the hypotheses that \( T(G) \) and \( CT(G) \) are finite respectively. For instance if \( t = IT(G) \) then \( G(s) \subsetneq G(t) = G \) for every \( s > t \). But Example 2.* is an example of a module \( G \) with infinite typeset such that \( IT(G) \notin T(G) \).

**Examples 4.6.** (1) Consider the module \( G = A_1(1, 0) + A_2(0, 1) + A_3(1, 1) \) in Example 2.28, as described in the beginning of this chapter. \( \text{OT}(G) \) consists of \( t_1 \lor t_2 \lor t_3 \) and \( t_i \lor t_j \) for \( i \neq j = 1, 2, 3 \). Then \( G[t_1 \lor t_2 \lor t_3] = 0 \), \( G[t_1 \lor t_2] = A_3 \), \( G[t_1 \lor t_3] = A_2 \), and \( G[t_2 \lor t_3] = A_1 \).

(2) Let \( G \) be the Pontryagin module (Example 1.47). If \( t \) is \( p \)-divisible then \( G[t] = 0 \), otherwise \( G[t] = G \).
PROOF: (1) If $\mu(G)$ is a rank-one homomorphic image of $G$ let $B_i = \mu(A_i(x,y))$ for $i = 1,2,3$ and $x, y = 0,1$ chosen appropriately so that $A_i(x,y) \subseteq G$. (Here, as previously, $A_i(x,y)$ denotes the set of elements $(ax,ay)$ for $a \in A_i$.) Then since $G = A_1(1,0) + A_2(0,1) + A_3(1,1), \mu(G) = B_1 + B_2 + B_3$ so that $\text{t}(\mu(G))$ is $t_1 \vee t_2, t_1 \vee t_3, t_2 \vee t_3$ or $t_1 \vee t_2 \vee t_3$, depending on whether one or none of the $B_i$ are trivial. Now $B_1$, for instance, is trivial if and only if $A_1(1,0) \subseteq \text{Ker} \mu$. Since $t_1 \vee t_1 \leq t_2 \vee t_3$ for $i = 1,2$, it follows that $\text{t}(\mu(G)) \leq t_2 \vee t_3 \iff A_1 \subseteq \text{Ker} \mu$ so that $G[t_2 \vee t_3] = A_1(1,0)$. And analogously for the other possibilities.

(2) $G$ is $p$-reduced and $p$-rank $G = 1 < \text{rank} G$. Thus if $H < G$ then $0 < p$-rank $H \leq p$-rank $G = 1$, so that $p$-rank $H = p$-rank $G$ and $p$-rank $G/H = 0$, so that $G/H$ is $p$-divisible. In particular, if $A$ is a rank-one module which is not $p$-divisible then $\text{Hom}(G,A) = 0$. Thus $G[t] = G$ for all types $t$ which are not $p$-divisible.

On the other hand, if $p'$ is a prime different from $p$ then $p'$-rank $G = rank G$, so that by Proposition 1.25 every rank-one homomorphic image of $G_{p'}$ is a free $W_{p'}$-module. It follows that if $G/H$ is a rank-one homomorphic image of $G$ then $G/H \approx p^{-\infty}$ and so $t(G/H) \leq t$ for any $p$-divisible type $t$. It then follows from Proposition 4.5 that $G[t] = 0$.

**FIX THIS!**

\[
\mathbf{PROPOSITION\ 4.7.}\ \begin{align*}
1) & \text{ If } s \leq t \text{ then } G(s) \supseteq G(t) \text{ and } G[s] \supseteq G[t]. \\
2) & \text{ } G(s \vee t) = G(s)(t) = G(s) \cap G(t). \\
3) & \text{ } G[s](t) = G[s] \cap G(t). \\
4) & \text{ If } T(G) \text{ is finite then } t \in T(G) \text{ if and only if } G(t) \supseteq G(s) \text{ for every } s > t. \\
5) & \text{ If } CT(G) \text{ is finite then } t \in CT(G) \text{ if and only if } G[t] \not\subset G[s] \text{ for every } s < t. \\
6) & \text{ If } s \not\leq t \text{ then } G(s) \not\subseteq G[t].
\end{align*}
\]

**PROOF:** (1) If $s \leq t$ and $g \in G(t)$ then $t(g) \geq s \geq t$ so that $g \in G(s)$.

Now let $A$ and $B$ be rank-one modules such that $t(A) = s$ and $t(B) = t$. If $s \not\leq t$ we may suppose WLOG that $A \subseteq B$. Then $\text{Hom}(G,A) \subseteq \text{Hom}(G,B)$ so if $g \in G[t]$ and thus $\gamma(g) = 0$ for all $\gamma \in \text{Hom}(G,B)$ then a fortiori $\gamma(g) = 0$ for all $\gamma \in \text{Hom}(G,A)$, so that $g \in G[s]$.

(2) By definition of $s \vee t$, for $g \in G$, $t(g) \geq s \vee t \iff t(g) \geq s \& t(g) \geq t$. Thus $G(s \vee t) = G(s) \cap G(t)$. Furthermore since $G(s) \not\simeq G, G(s)(t) = G(s) \cap G(t)$ by Proposition 4.4.

(3) Since $G[s]$ is a pure submodule of $G$, $G[s](t) = G[s] \cap G(t)$ by Proposition 4.4.

(4) If $t \in T(G)$ then there exists $g \in G$ with $t(g) = t$, and so $g \in G(t)$ but $g \not\in G(s)$ for $s > t$. Thus $G(t) \not\subset G(s)$. Conversely if $t \not\in T(G)$ then no element of $G$ has type $t$, so if $g \in G(t)$ then $t(g) > t$, which means that $QG(t) = \bigcup_{s > t} QG(s)$, where the union can be taken over $s$ with $s \in T(G)$ and $s > t$. If $T(G)$ is finite then since a vector space cannot be a finite union of proper subspaces we conclude that $QG(t) = QG(s)$ for some $s > t$. Since $G(t)$ and $G(s)$ are pure submodules of $G$, we conclude that $G(t) = G(s)$.

(5) ( $\iff$ ): If $G(t) \not\subseteq G[s]$ for every $s < t$ then by Proposition 4.5, $\text{rank } \text{Hom}(G,B) < \text{rank } \text{Hom}(G,A)$ so $Q\text{Hom}(G,B) \not\subseteq Q\text{Hom}(G,A)$ whenever $A$ is a rank-one module with $t(A) = t$ and $B$ a rank-one module with $t(B) = s < t$. Now if
CT(G) is finite, then there are only finitely many B with t(B) ∈ CT(G) and t(B) < t
(Explain!). Since a vector space over the infinite field Q cannot be a union of a finite
number of proper subspaces, thus there exists φ ∈ Hom(G, A) with φ ∉ QHom(G, B) for
any B with t(B) < t. Thus t(φ(A)) = t so t ∈ CT(G).

(⇒): If t ∈ CT(G) and s < t and B ⊆ A are rank-one modules with t(A) = t,
t(B) = s, then there exists φ ∈ Hom(G, A) with φ(G) = A. Thus φ ∉ Hom(G, B) so
that Hom(G, B) ⊆ Hom(G, A). Thus by Proposition 4.6 rank G[t] < rank G[s] so that
G[t] ⊈ G[s].

(6) If s ⊈ t and A is a rank-one module with t(A) = t and φ ∈ Hom(G, A) then
A(s) = 0 and so by Proposition 4.4 φ(G(s)) ⊆ A(s) = 0. Thus G(s) ⊆ G[t].

DOMINATION. We say that H dominates G if IT(H) ≥ OT(G).

PROPOSITION 4.8. (1) H dominates G if and only if QHom(G, H) = Hom(QG, QH).
(2) If H dominates G then G is isomorphic to a submodule of H^r, for any r such
that r(rank H) ≥ rank G.
(3) Domination is transitive, i.e. if K dominates H and H dominates G, then K
dominates G.

PROOF: (1) Since Hom(QG, QH) is generated by those linear transformations
QG → QH with one-dimensional image, QHom(G, H) = Hom(QG, QH) if and
only if QHom(G, H) contains every φ: QG → QH with dim φ(QG) = 1, i.e. if and
only if for every φ: G → QH with rank φ(G) = 1 there exists w ≠ 0 ∈ W with
wφ(G) ⊆ H. The latter condition holds if and only if t(φ(G)) ≤ t(H ∩ φ(QG)). So
QHom(G, H) = Hom(QG, QH) ⇐⇒ the type of every rank-one homomorphic image of
G is less than or equal to the type of every pure rank-one submodule of H, i.e. s ≤ t for
all s ∈ CT(G), t ∈ T(H). But this is equivalent to saying OT(G) ≤ IT(H). (Rewrite
this!)

(2) If r(rank H) ≥ rank G then dim QG ≤ dim QH^r, so QG is isomorphic to a
subspace of QH^r. If H dominates G then clearly H^r also dominates G and so if
φ: QG → QH^r is a monomorphism then φ ∈ QHom(G, H^r). Then for some w ≠ 0 ∈ W,
wφ is a monomorphism from G into H^r.

(3) Recall from Proposition 2.33 that IT(H) ≤ OT(H). Thus if K dominates H
and H dominates G then OT(G) ≤ IT(H) ≤ OT(H) ≤ IT(K), so that K dominates
G. ✷

PROPOSITION 4.9. Let H dominate G and let p be a prime. Then

(1) Either H is p-divisible or G_p is a free W_p-module.
(2) p-rank Hom(G, H) = (p-rank)G(p-rank)H.
(3) Hom(G, H)_p = Hom(G_p, H_p).

PROOF: (1) If H is not p-divisible then IT(H) is not p-divisible. If H dominates
G, then OT(G) ≤ IT(H), so OT(G) is not p-divisible. Thus by Proposition 2.33
p-rank G = rank G so that G_p is a free W_p-module by Proposition 1.25.
(3) By Proposition 1.39, Hom\((G, H)_p \vartriangleleft \text{Hom}(G_p, H_p)\). But since \(H\) dominates \(G\), Hom\((QG, QH) = Q\text{Hom}(G, H) \subseteq Q\text{Hom}(G_p, H_p) \subseteq \text{Hom}(QG, QH)\), so that Hom\((G, H)\) is also an essential submodule of Hom\((G_p, H_p)\). Hence by Proposition 1.17 they are identical.

(2) By Proposition 1.24, \(\text{p-rank Hom}(G, H) = \text{p-rank Hom}(G, H)_p\). By (3) Hom\((G, H)_p = \text{Hom}(G_p, H_p)\). Thus by (1) we need consider only the case where \(G_p\) is a free \(W_p\)-module or the case where \(H_p\) is divisible. In the first case \(\text{Hom}(G_p, H_p) \approx H_p^r\), with \(r = \text{rank } G = \text{p-rank } G\), and in the second case by Corollary 1.34 \(\text{Hom}(G_p, H_p)\) is also divisible, so that p-rank \(\text{Hom}(G, H) = 0\). In either case, the result is clear. \(\square\)

Proposition 4.9 shows that if \(G\) and \(H\) are p-local modules then \(H\) dominates \(G\) if and only if either \(H\) is divisible or \(G\) is a free \(W_p\)-module. Thus domination is not an interesting concept for p-local modules.

**Proposition 4.10.** Let \(K\) dominate \(G\).

1. If \(G' \vartriangleleft G\) then the restriction map \(\text{Hom}(G, K) \twoheadrightarrow \text{Hom}(G', K)\) is a surjection.
2. If \(\varphi : K \rightarrow K'\) is a surjection, then \(\varphi_* : \text{Hom}(G, K) \rightarrow \text{Hom}(G, K')\) is surjective.
3. If \(\varphi : K \twoheadrightarrow G\) is a surjection, then \(\varphi\) splits.
4. If \(K \triangleleft G\) then \(K\) is a summand of \(G\).

**Proof:** (1) By Proposition 0.* it suffices to prove that for each prime \(p\) the map \(\text{Hom}(G, K)_p \rightarrow \text{Hom}(G', K)_p\) is surjective. But by Proposition 4.9 we can write this as \(\text{Hom}(G_p, K_p) \rightarrow \text{Hom}(G_p', K_p)\). Furthermore by Proposition 4.9 either \(K_p\) is divisible, hence injective, in which case clearly this map is a surjection, or \(G_p\) is projective, in which case \(G_p/G_p'\) is also projective and \(G_p'\) is a summand of \(G_p\) and once again the map is a surjection.

(2) Analogous. It suffices to prove that for each \(p\) the map \(\text{Hom}(G_p, K_p') \rightarrow \text{Hom}(G_p', K_p')\) is surjective. This is clear if \(G_p\) is free. On the other hand, if \(K_p\) is divisible then so is \(K_p'\) and the map in question simply becomes \(\text{Hom}(QG, QK) \rightarrow \text{Hom}(QG, QK')\), which is a surjection by vector space theory.

(3) Applying (2) with \(K' = G\) we see that the induced map \(\varphi_* : \text{Hom}(G, K) \rightarrow \text{Hom}(G, G)\) is surjective. In particular, \(1_G = \varphi_*(\delta) = \varphi\delta\) for some \(\delta \in \text{Hom}(G, K)\), so that \(\varphi\) splits.

(4) Applying (1) with \(G' = K\) we see that the restriction map \(\text{Hom}(G, K) \rightarrow \text{Hom}(K, K)\) is surjective, so that there exists \(\gamma \in \text{Hom}(G, K)\) so that the restriction of \(\gamma\) to \(K\) is the identity. Thus \(\gamma\) is a projection from \(G\) onto \(K\). \(\square\)

It needs to be noted that parts (3) and (4) of Proposition 4.10 are stated in somewhat fraudulent generality. If \(G\) is a homomorphic image of \(H\) and \(H\) dominates \(G\) then \(G\) dominates itself, since \(\text{OT}(G) \leq \text{OT}(H) \leq \text{IT}(G)\). Likewise if \(H \vartriangleleft G\) and \(H\) dominates \(G\) then \(H\) dominates itself. But we will see shortly that the only modules which dominate themselves are those which are direct sums of mutually quasi-isomorphic rank-one modules (see Corollary 4.15 below).
PROPOSITION 4.11. If $H$ dominates $G$ then for any finite rank torsion free module $K$, $H \otimes K$ dominates $G$ and $\text{Hom}(G, H \otimes K) \approx \text{Hom}(G, H) \otimes K$.

PROOF: (1) By Proposition 2.35 $\text{IT}(H \otimes K) = \text{IT}(H) \text{IT}(K) \geq \text{IT}(H) \geq \text{OT}(G)$.

(2) By Proposition 1.40 the canonical map $\text{Hom}(G, H) \otimes K \rightarrow \text{Hom}(G, H \otimes K)$ is a monomorphism with pure image. But since $H$ dominates $G$, 

\[ \text{rank} \text{Hom}(G, H) \otimes K = (\text{rank} G)(\text{rank} H)(\text{rank} K) \geq \text{rank} \text{Hom}(G, H) \otimes K. \]

Therefore this map must also be surjective.

\[ \checkmark \]

**t-SATURATED AND t-BOUNDED MODULES.** We say that $G$ is **t-saturated** if $G(t) = G$ and **t-bounded** if $G[t] = 0$. We say that $G$ is **t-projective** if it is completely decomposable and homogeneous of type $t$. In other words, $G$ is t-projective if $G = \bigoplus A_i$, where the $A_i$ are rank-one modules with $t(A_i) = t$.

**PROPOSITION 4.12.** (1) Pure submodules and homomorphic images of t-saturated modules are t-saturated. Furthermore if $G$ is an essential submodule of $H$ and $G$ is t-saturated then $H$ is t-saturated.

(2) Submodules and homomorphic images of t-bounded modules are t-bounded.

(3) A module quasi-isomorphic to a t-saturated module is t-saturated and a module quasi-isomorphic to a t-bounded module is t-bounded.

(4) Let $H$ be a pure submodule of $G$. Then $G/H$ is t-bounded $\iff H \supset G[t]$.

(5) If $G_1, \ldots, G_n$ are t-saturated submodules of $G$, then $G_1 + \cdots + G_n$ is t-saturated.

(6) If $G_1, \ldots, G_n$ are t-bounded submodules of $G$, then $G_1 + \cdots + G_n$ is t-bounded.

PROOF: (1) If $G$ is t-saturated and $\varphi : G \rightarrow K$ is surjective then by Proposition 4.4

\[ K(t) \supseteq \varphi(G(t)) = \varphi(G) = K, \text{ so } K \text{ is t-saturated.} \]

And if $H \vartriangleleft G$ then by Proposition 4.4 $H(t) = H \cap G(t) = H \cap G = H$, so $H$ is t-saturated. And if $G$ is an essential submodule of $H$ then $G = G(t) \subseteq H(t)$, so that $H(t)$ is an essential submodule of $H$. Since $H(t) \vartriangleleft H$, $H = H(t)$, so $H$ is t-saturated.

(2) By Proposition 2.33 if $H \subseteq G$ then $\text{OT}(H) \leq \text{OT}(G)$ and if $H \vartriangleleft G$ then $\text{OT}(G/H) \leq \text{OT}(G)$. It then follows from (?next proposition?) that if $G$ is bounded then so are $H$ and $G/H$.

(3) If $G$ and $H$ are quasi-isomorphic and $G$ is t-saturated then we may assume that $G$ is an essential submodule of $H$. Therefore by (1) $H$ is t-saturated. Likewise if $H$ is t-bounded then by (5) $G$ is t-bounded.

(4) ($\Leftarrow$): Let $A$ be a rank-one module with $t = t(A)$. Since $\text{Hom}(G/G[t], A) \approx \text{Hom}(G, A)$, by Proposition 4.5 

\[ \text{rank} \text{Hom}(G/G[t], A) = \text{rank} \text{Hom}(G, A) = \text{rank} G[G[t], Q] = \text{rank} \text{Hom}(G/G[t], QA), \]

so that by Proposition 4.8 $A$ dominates $G/G[t]$, so $\text{OT}(G/G[t]) \leq \text{IT}(A) = t$. But if $H \supseteq G[t]$ then $G/H$ is a homomorphic image of $G/G[t]$ so by Proposition 2.33, $\text{OT}(G/H) \leq \text{OT}(G/G[t]) \leq t$ and so $G/H$ is t-bounded.

($\Rightarrow$): If $G/H$ is t-bounded let $g \in G[t]$. Let $A$ be a rank-one module with $t(A) = t$ and $\varphi : G/H \rightarrow A$. Then $\varphi$ induces $\varphi : G \rightarrow A$ and since $g \in G[t]$, then $\varphi(g) = 0$. Thus $\varphi(g + H) = 0$. Since this is true for every $\varphi$ mapping $G$ into $A$ then $g + H \in (G/H)[t] = 0$, i.e. $g \in H$. Thus $G[t] \subseteq H$. 

(5) & (6) $G_1 + \cdots + G_n$ is a homomorphic image of $G_1 \oplus \cdots \oplus G_n$, which is $t$-saturated or $t$-bounded if each $G_i$ is. Thus the result follows from (1) and (2).

**Proposition 4.13.** (1) $G$ is $t$-saturated if and only if $\text{IT}(G) \geq t$. 
(2) $G$ is $t$-bounded if and only if $\text{OT}(G) \leq t$.
(3) If $A$ is a rank-one module with $t = t(A)$ then $G$ is $t$-saturated if and only if $G$ dominates $A$ and $G$ is $t$-bounded if and only if $A$ dominates $G$.
(4) A module is $t$-saturated if and only if it is a homomorphic image of a (possibly infinite) direct sum of rank-one modules of type $t$.
(5) A module is $t$-bounded if and only if it is isomorphic to a submodule of a $t$-projective module.
(6) If $t$ is not $p$-divisible and $G$ is $t$-bounded then $p$-rank $G = \text{rank} G$ and $G_p$ is a free $W_p$-module.
(7) If $G$ is $t$-bounded and $H$ $t$-saturated, then $H$ dominates $G$.

**Proof:** (1) $G$ is $t$-saturated if and only if $G = G(t)$, i.e. if and only if $t(g) \geq t$ for all $g \in G$. This is true if and only if $\text{IT}(G) \geq t$.
(2) ($\Leftarrow$): Suppose $\text{OT}(G) \leq t$ and let $g \neq 0 \in G$. Then there exists a map $\varphi: G \to Q$ with $\varphi(g) \neq 0$. Since $\text{OT}(G) \leq t$, $t(\varphi(G)) \leq t$. Thus there is a non-trivial map from $G$ into the rank-one module $\varphi(G)$ with $t(\varphi(G)) \leq t$, so $g \notin G[t]$. Thus $G[t] = 0$, so $G$ is $t$-bounded.
(3) This is just a restatement of (1) and (2).
(4) Let $G$ be $t$-saturated. Then $G$ is a homomorphic image of $\bigoplus \{A \subseteq G \mid \text{rank} A = 1 \text{ and } t(A) = t\}$. The converse follows from Proposition 4.12 (1).

(5) ($\Rightarrow$): By (3) $G$ is $t$-bounded if and only if $A$ dominates $G$, where $t = t(A)$. Thus by Proposition 4.8 $G$ is isomorphic to a submodule of $A^r$, i.e. to a submodule of a $t$-projective module of rank $r$.
(6) If $G$ is $t$-bounded and $t$ is not $p$-divisible, then $\text{OT}(G)$ is not $p$-divisible since $\text{OT}(G) \leq t$. Thus by Proposition 2.33 $p$-rank $G = \text{rank} G$. Then by Proposition 1.25, $G_p$ is a free $W_p$-module.
(7) If $G$ is $t$-bounded and $H$ $t$-saturated, then $\text{OT}(G) \leq t \leq \text{IT}(H)$. Hence $H$ dominates $G$.

The following Proposition will partially justify the use of the term “$t$-projective.”
PROPOSITION 4.14.  Let $A$ be a subring of $Q$, let $t = t(A)$, and let $G$ is any finite rank torsion free module.

(1) $G$ is $t$-saturated if and only if $G$ is isomorphic to an $A$-module.
(2) $G$ is $t$-projective if and only if $G$ is isomorphic to a projective $A$-module.

PROOF: (1) If $G$ is an $A$-module and $g \neq 0 \in G$ then $Ag \subseteq G \cap Qg$ and since $Ag \cong A$ we see that $t(A) \subseteq t(G \cap Qg) = t(g)$, so $g \in G(t)$. Thus $G$ is $t$-saturated. Conversely if $G$ is $t$-saturated then we will show that $G$ is an $A$-submodule of $QG$, i.e that $AG \subseteq G$. If $g \neq 0 \in G$ then $Ag \subseteq QG$ and $Ag \cong A$. Then $t(G \cap Ag) = t(G \cap Qg \cap Ag) = t(g) \wedge t(A) = t(A)$ since $t(g) \geq t(A)$. Thus by Proposition 2.15, $G \cap Ag$ is a finitely generated $A$-submodule of $Qg$, and hence of $QG$. Thus $Ag \subseteq A(G \cap Ag) \subseteq G \cap Ag \subseteq G$. Since this holds for all $g \in G$, $G$ is an $A$-submodule of $QG$.

(2) By Proposition 0. A is a dedekind domain. (Recall that by convention the word “subring” means a $W$-subalgebra, i.e. $A$ is a subring of $Q$ containing $W$.) Thus an $A$-module is projective if and only if it is a direct sum of $A$-ideals. On the other hand, $G$ is $t$-projective if and only if $G$ is a direct sum of rank-one modules of type $t$. But by Proposition 2.15 a rank-one module has type $t$ if and only if it is isomorphic to a fractional ideal in $A$, and hence to a projective $A$-module. Hence $G$ is $t$-projective if and only if $G$ is isomorphic to a projective $A$-module.  

EXAMPLES 4.15. (1) It $t = t(W)$ then for any $G$, $G(t) = G$ and $G[t]$ is a summand of $G$ and is the smallest pure submodule $H$ of $G$ such that $G/H$ is projective. Every $G$ is $t$-saturated and $G$ is $t$-bounded if and only if it is projective.

(2) If $t = t(Q)$ then for any $G$, $G[t] = 0$ and $G(t) = d(G)$. Every $G$ is $t$-bounded and $G$ is $t$-saturated if and only if it is divisible.

PROOF: By Proposition 2.32 $IT(G) \geq t(W)$ and $OT(G) \leq t(Q)$ for every $G$, and $OT(G) = t(W)$ if and only if $G$ is projective and $IT(G) = t(Q)$ if and only if $G$ is divisible. Thus since by Proposition 4.12 (6) $G[t(W)]$ is the maximal pure submodule $H$ of $G$ such that $G/H$ is $t$-bounded, it follows that it is the maximal pure submodule such that $G/H$ is projective. Since $G/G[t]$ is projective, $G[t]$ is a summand.

Likewise $G(t(Q))$ is the maximal $t(Q)$-saturated submodule of $G$, i.e. the maximal divisible submodule of $G$.

BAER’S LEMMA. What makes everything work is the following theorem, due to Reinhold Baer. It is essentially just a special case of Proposition 4.10, however it seems worthwhile to also show how the traditional proof can be adapted to the dedekind domain case.

THEOREM 4.16. If $H \triangleleft G$ and $G/H$ is $t$-bounded and $G = H + G(t)$, then $G = H \oplus L$ for some $L \subseteq G(t)$.

FIRST PROOF. $G(t)/(G(t) \cap H) \cong (G(t) + H)/H = G/H$. Now since $G(t)$ is $t$-saturated and $G/H$ is $t$-bounded, $G(t)$ dominates $G/H$. Thus by Proposition 4.10
For this proof, we need to assume that $G/H$ is $t$-projective. It turns out that this involves no loss of generality, since the result in this case is adequate to prove Corollary 4.14 below, which implies that if $G/H$ is $t$-bounded then it must be $t$-projective. Assume then that $G/H = \bigoplus A_i$, where the $A_i$ are rank-one modules of type $t$. Let $\varphi: G \to G/H$ be the quotient map. We need to find $\sigma: G/H \to G$ splitting $\varphi$. It suffices to find $\sigma_i: A_i \to G$ for each $i$ such that $\varphi \sigma_i = 1_{A_i}$.

Now fix $i$. Since $G = G(t) + \ker \varphi$, there exists a pure rank-one submodule $B$ of $G(t)$ with $0 \neq \varphi(B) \subseteq A_i$. Then $t(B) \geq t$ and also since $\text{Hom}(B, A_i) \neq 0$, $t(B) \leq t$. So $t(\varphi(B)) = t(B) = t = t(A_i)$. Thus by Proposition 2.2, $A_i/\varphi(B)$ is cyclic. Let $a \in A_i$ be a pre-image for a generator of $A_i/\varphi(B)$. Since $A \subseteq \varphi(G(t))$ there exists a pure rank-one submodule $C$ of $G(t)$ with $a \in \varphi(C)$. Thus $\varphi(B) + \varphi(C) = A_i$. The restrictions of $\varphi$ to $B$ and $C$ are not trivial, hence are monic by Proposition 2.*, so there exist $\sigma_B: \varphi(B) \to B$ and $\sigma_C: \varphi(C) \to C$ such that $\varphi \sigma_B = 1_B$ and $\varphi \sigma_C = 1_C$. Now by Proposition 2.* $[A_i: A_i] = [\varphi(B) + \varphi(C): A_i] = [\varphi(B): A_i] + [\varphi(C): A_i]$, so there exist in particular $u \in [\varphi(B): A_i]$ and $v \in [\varphi(C): A_i]$ such that $u + v = 1$. Now define $\sigma_i: A_i \to G$ by $\sigma_i(x) = \sigma_B(ux) + \sigma_C(vx)$. Note that by construction $ux \in \varphi(B)$ and $vx \in \varphi(C)$, and furthermore $\sigma_i(x) = \varphi \sigma_B(ux) + \varphi \sigma_C(vx) = ux + vx = x$. I.e. $\varphi \sigma_i = 1_{A_i}$.

Now combining these maps $\sigma_i$ yields a map $\sigma: G/H = \bigoplus A_i \to G$ such that $\varphi \sigma = 1_{G/H}$. Thus $\varphi$ splits and $H$ is a summand of $G$.  

\textbf{Proposition 4.17.} A finite rank torsion free module $G$ is both $t$-saturated and $t$-bounded if and only if $G$ is $t$-projective.

\textbf{Proof:} ($\Leftarrow$): Clear.

($\Rightarrow$): By induction on rank $G$. Suppose that $G$ is both $t$-saturated and $t$-bounded. If $G$ has rank-one we see that $t(G) = t$ and $G$ is trivially $t$-projective. If rank $G > 1$, let $H < G$ be such that rank $G/H = 1$. Then by Proposition 4.12 $G/H$ is $t$-bounded and so by Baer’s Lemma, $G \approx H \oplus G/H$. By induction, $H$ is $t$-projective. Therefore $G$ is $t$-projective.

\textbf{Corollary 4.18.} (1) If $G$ and $H$ are finite rank torsion free modules then $G$ and $H$ mutually dominate each other if and only if there exists a type $t$ such that $G$ and $H$ are each $t$-projective.

(2) If $G$ and $H$ are $t$-projective modules and $QG = QH$ then $G$ and $H$ are quasi-equal.

(3) A finite rank torsion free module $G$ dominates itself if and only if $G$ is $t$-projective for some type $t$.

\textbf{Proof:} (1) $G$ and $H$ mutually dominate each other if and only if $\text{IT}(G) \leq \text{OT}(G) \leq \text{IT}(H) \leq \text{OT}(H) \leq \text{IT}(G)$ (c.f. Proposition 2.33),
i.e. if and only if $G$ and $H$ are both $t$-saturated and $t$-bounded, where
$t = \text{IT}(G) = \text{IT}(H) = \text{OT}(G) = \text{OT}(H)$. By Proposition 4.17 this is equivalent to $G$ and $H$ both being $t$-projective.

(2) If $G$ and $H$ are $t$-projective and $QG = QH$, then $G$ and $H$ dominate each other by (1). Thus $1_{QG}$ belongs to both $\text{QHom}(G, H)$ and $\text{QHom}(H, G)$. Thus $G$ and $H$ are quasi-equal.

(3) This follows by applying (1) to the case $H = G$. ✓

PROPOSITION 4.19. (1) Pure submodules and torsion free homomorphic images of $t$-projective modules are $t$-projective.
(2) A pure submodule of a $t$-projective module is a direct summand.
(3) A module quasi-isomorphic to a $t$-projective module is $t$-projective.
(4) A $t$-projective submodule $H$ of a $t$-projective module $G$ is a quasi-summand of $G$.

PROOF: (1) By Proposition 4.12 a pure submodule or homomorphic image of a $t$-projective module is both $t$-saturated and $t$-bounded. Thus the result follows from Proposition 4.17.

(2) If $G$ is $t$-projective and $H \triangleleft G$, then by Proposition 4.17 $G$ is $t$-saturated and by Proposition 4.12 $G/H$ is $t$-bounded. Thus $H$ is a summand of $G$ by Baer's Lemma (Theorem 4.16).

(3) If $G$ is $t$-projective and $H \sim G$, the $H$ is $t$-bounded and $t$-saturated, by Proposition 4.17 $H$ is $t$-projective.

(4) By (2) $H_*$ is a summand of $H$ and hence by (1) is $t$-projective. Since $H$ and $H_*$ have the same rank, it is thus clear that they are quasi-isomorphic. Thus by Proposition 3.9 the inclusion map $H \hookrightarrow H_*$ is a quasi-isomorphism. Hence since $H_*$ is a summand of $G$, $H$ is a quasi-summand. ✓

The following proposition is extremely useful for avoiding calculations when constructing examples:

PROPOSITION 4.20. If $H \triangleleft G$ and $G = H + K$ where $K$ is $t$-projective for some type $t$, then there exists a pure submodule $K'$ of $K$ such that $G = H \oplus K'$.

PROOF: We can apply Baer's Lemma. In fact, since $K \subseteq G(t)$, clearly $G = H + G(t)$ and $G/H$ is $t$-projective since it is a homomorphic image of $K$. Thus by Baer's Lemma $H$ is a summand of $G$. Now we need to see that the complementary summand can be chosen to be a submodule of $K$. One can verify this rather tediously by retracing the proof of Baer's Lemma. A faster way is to note that by Baer's Lemma $K = (H \cap K) \oplus K'$ for some $K' \subseteq K$. As in the proof of Theorem 4.16 one then sees that $G = H \oplus K'$. ✓
EXAMPLE 4.21. Consider again Example 1.49. In this example \( H \) is generated by \( B \oplus C \) together with \((b+c)/w\) where \( B \) and \( C \) are rank-one modules with incomparable types, and \( b \in B \), \( c \in C \) are such that \( b \notin wB \), \( c \notin wC \). We let \( A \) be a module isomorphic to \( B \) and let \( a \in A \) correspond to \( b \in B \) under some isomorphism. Now let \( B_2 \) be the pure rank-one submodule of \( A \oplus H \) generated by \( a + 2b \), and let \( H_2 \) be the purification of \( B_2 + C \). Then there is an isomorphism \( \theta \) from \( H_2 \) to the submodule of \( QB \oplus QC \) generated by \( B \oplus 2C \) together with \((b+2c)/w\), given by \( \theta(a + 2b) = b \) and \( \theta(2c) = c \). We have seen that under suitable hypotheses on \( A \) and \( B \), \( H \) and \( H_2 \) are not isomorphic. But there exists a rank-one submodule \( A_2 \) of \( A \oplus H \) such that \( A_2 \oplus H_2 = A \oplus H \), and if \( W \) is a principal ideal domain then \( A_2 \approx A \).

PROOF: Let \( G = A \oplus H \). Clearly \( G = H_1 + (A \oplus B) \). Since \( A \oplus B \) is \( t(A) \)-projective, by Proposition 4.20 \( A \oplus H = A_2 \oplus H_2 \) for some rank-one submodule \( A_2 \) of \( A \oplus B \). By Proposition 4.19 \( A_2 \) is \( t \)-projective, i.e. \( t(A_2) = t(A) \). Thus if \( W \) is a principal ideal domain then \( A_2 \approx A \) by Corollary 2.10. \( \checkmark \)

In contrast to most of the theorems in this chapter and the next, the dual theorem to Baer’s Lemma is not valid. The dual theorem would assert that if \( H \) is a \( t \)-projective pure submodule of \( G \) such that \( H \cap G[t] = 0 \) then \( H \) is a direct summand. We have the following simple counter-example:

EXAMPLE 4.22. Consider again Example 1.49. In this example \( H \) is generated by \( B \oplus C \) together with \((b+c)/w\) where \( B \) and \( C \) are rank-one modules with incomparable types, and \( b \in B \), \( c \in C \) are such that \( b \notin wB \), \( c \notin wC \). Now let \( t = t(B) \). Then \( H[t] = C \) and \( B \cap H[t] = 0 \). But \( B \) is not a direct summand of \( H \).

PROOF: If \( \varphi \in \text{Hom}(H,B) \) then \( \varphi(C) = 0 \) since \( t(C) \) and \( t \) are incomparable. Thus \( C \subseteq H[t] \). On the other hand, since \( H/C \approx B \), \( H[t] \subseteq C \) by Proposition 4.12. We have seen earlier that \( H \) is indecomposable, so \( B \) is not a summand of \( H \). \( \checkmark \)

However we do have the following two propositions, which are slightly weaker.

PROPOSITION 4.23. Let \( G \) be \( t \)-bounded and \( H \) a pure submodule of \( G \). Let \( K \) be a \( t \)-saturated module. Then every homomorphism \( \varphi: H \to K \) extends to a homomorphism \( G \to K \).

PROOF: If \( G \) is \( t \)-bounded and \( K \) \( t \)-saturated then \( K \) dominates \( G \). Hence this is a special case of Proposition 4.10. \( \checkmark \)

PROPOSITION 4.24. (1) If \( H \) is a \( t \)-saturated submodule of \( G \) and \( H \cap G[t] = 0 \) then \( H \) is \( t \)-projective and \( G \) is quasi-equal to \( H \oplus L \) for some submodule \( L \) with \( G[t] \subseteq L \) and \( H \oplus G[t] \) is a quasi-pure submodule of \( G \).

(2) If in addition \( H + G[t] \) is a pure submodule of \( G \) then \( G = H \oplus L \) for some \( L \) with \( G[t] \subseteq L \).

(3) If \( G \) is \( t \)-bounded and \( H \) is a pure \( t \)-saturated submodule of \( G \), then \( H \) is a \( t \)-projective direct summand of \( G \).
PROOF: (3) If $G$ is $t$-bounded and $H$ $t$-saturated, then Proposition 4.23 applied to $1_H$ shows that $H$ is a summand of $G$. Since $H$ is both $t$-saturated and $t$-bounded, it is $t$-projective by Proposition 4.17.

(2) If $H + G[t] = H \oplus G[t]$ is a pure submodule of $G$ then $(H \oplus G[t])/G[t]$ is a pure $t$-saturated submodule of the $t$-bounded module $G/G[t]$, and hence is a summand of $G/G[t]$ by (3). Thus there is a projection $G/G[t] \to (H \oplus G[t])/G[t]$ which leads to a sequence of maps

$$G \to G/G[t] \to (H \oplus G[t])/G[t] \cong H.$$  

Let $\theta$ be the composition of these maps. Then $\theta$ restricts to the identity on $H$, showing that $G = H \oplus \ker \theta$. Note that $G[t] \subseteq \ker \theta$.

(1) $(H \oplus G[t])/G[t]$ is $t$-bounded since it is a submodule of $G/G[t]$ and is $t$-saturated since it is isomorphic to $H$. And $(H \oplus G[t])_*/G[t]$ is also $t$-bounded and $t$-saturated. Hence these are both $t$-projective and hence quasi-isomorphic, since they have the same rank. Thus by Proposition 3.9 they are quasi-equal, so $G/G[t]$ is a quasi-submodule of $G$. Thus by Proposition 3.14 $H \oplus G[t]$ is a quasi-submodule of $G$. Now the reasoning in (2) applies, except that in this case we can only conclude that $\theta$ is a quasi-projection of $G$ “quasi-onto” $H$. Thus $G$ is quasi-equal to $H \oplus L$ for some $L$ with $G[t] \subseteq L$. $\Box$

PROPOSITION 4.25. If $t = OT(G)$ then $G(t)$ is $t$-projective and is a summand of $G$.

PROOF: Since $t = OT(G)$, $G$ is $t$-bounded. Since $G(t) \vartriangleleft G$, by Proposition 4.12 $G(t)$ is $t$-bounded. Since it is $t$-saturated, it is $t$-projective by Proposition 4.17 and is a summand of $G$ by Proposition 4.24. $\Box$

PROPOSITION 4.26. Let $G$ be a finite rank torsion free module.

1. $G(t) + G[t] = G_t \oplus G[t]$, where $G_t$ is a $t$-projective module.
2. $G(t) + G[t]$ is a quasi-pure submodule of $G$.
3. $G_t$ is a maximal $t$-projective quasi-summand of $G$.
4. $(G(t) + G[t])_*/G[t]$ is a $t$-projective direct summand of $G/G[t]$.
5. $G_t \sim (G(t) + G[t])_*/G[t]$.

PROOF: (1) $(G(t) + G[t])/G[t]$ is $t$-bounded since it is a submodule of the $t$-bounded module $G/G[t]$. Hence by Theorem 4.16 $G[t]$ is a summand of $G(t) + G[t]$ and the complementary summand is $t$-projective.

(2) By Proposition 4.24 $G_t \oplus G[t] = G(t) + G[t]$ is quasi-pure in $G$.

(3) Since $G_t$ is $t$-saturated and $G_t \cap G[t] = 0$, by Proposition 4.24 $G_t$ is a quasi-summand of $G$. Clearly it is a maximal $t$-projective quasi-summand, since if $H$ is a $t$-projective quasi-summand of $G$ then $H \subseteq G(t)$ and if $\pi: G \to H$ is a quasi-projection then $\pi(G[t]) \subseteq H[t] = 0$, so that $H \cap G[t] = 0$.

(4) $(G(t) + G[t])_*/G[t]$ is a $t$-saturated submodule of the $t$-bounded module $G/G[t]$. Therefore it is a $t$-projective direct summand of $G/G[t]$ by Proposition 4.24.

(5) $G(t) + G[t]$ is quasi-equal to $(G(t) + G[t])_*$ by (2), so that $G_t \sim (G(t) + G[t])/G[t] \sim (G(t) + G[t])_*/G[t]$. $\Box$
**COMPLETELY DECOMPOSABLE MODULES.** Recall that a module is called **completely decomposable** if it is a direct sum of rank-one modules. We can now rephrase this by saying that a module is completely decomposable if it is a direct sum of \( t \)-projective modules for various \( t \).

By Jónsson’s Theorem (Theorem 3.24) a maximal completely decomposable quasi-summand of a module \( G \) is determined uniquely up to quasi-isomorphism. It is somewhat surprising, although quite elementary, that the same is true of a maximal completely decomposable actual summand of \( G \).

**PROPOSITION 4.27.** (1) A maximal \( t \)-projective summand of a finite rank torsion free module \( G \) is unique up to quasi-isomorphism.

(2) A maximal completely decomposable summand of \( G \) is unique up to quasi-isomorphism.

**PROOF:** (1) Let \( G = C \oplus H = D \oplus K \), where \( C \) and \( D \) are \( t \)-projective and \( H \) and \( K \) have no non-trivial \( t \)-projective summand. Then \( H \cap D \subsetneq D \) so by Proposition 4.19 \( H \cap D \) is a summand of \( D \), and hence a summand of \( G \), and hence a summand of \( H \). But by assumption \( H \) has no non-trivial \( t \)-projective summand. Thus \( H \cap D = 0 \), so the projection of \( G \) onto \( C \) restricts to a monomorphism from \( D \) into \( C \). Likewise there is a monomorphism from \( C \) into \( D \), so that by Proposition 3.1 \( C \) and \( D \) are quasi-isomorphic.

(2) Let \( G = H \oplus \bigoplus C_t \) where each \( C_t \) is \( t \)-projective and \( H \) has no non-trivial completely decomposable summand. Then in particular for each \( t \), \( H \) has no \( t \)-projective summand so by (1) each \( C_t \) is unique up to quasi-isomorphism. Thus \( \bigoplus C_t \) is unique up to quasi-isomorphism. \( \checkmark \)

Note that if \( W \) is a principal ideal domain then . . .

In light of Proposition 4.27 it is very plausible to infer think that if \( C \) is a maximal completely decomposable summand of \( G \) and \( D \) a maximal completely decomposable summand of \( H \), then \( C \oplus D \) is a maximal completely decomposable summand of \( G \oplus H \). However, this need not be the case. In fact, as the following example shows, it is possible for \( G \) and \( H \) to both have no non-trivial completely decomposable summand at all and yet for \( G \oplus H \) to have a very large one.

**EXAMPLE 4.28.** Let \( A_1, \ldots, A_n \) be rank-one modules with mutually incomparable types, and for each \( i \) let \( B_i \) be a rank-one module with \( B_i \cong A_i \). Suppose further that there exist distinct primes \( p \) and \( q \) such that none of the \( A_i \) is \( p \)-divisible or \( q \)-divisible, and for each \( i \) choose \( a_i \in A_i \) with \( a_i \notin pA_i \) and \( b_i \in B_i \) with \( b_i \notin qB_i \). Let \( G \) be the submodule of \( QA_1 \oplus \cdots \oplus QA_n \) generated by \( A_1 \oplus \cdots \oplus A_n \) together with \( p^{-1}(a_1 + \cdots + a_n) \) and let \( H \) be the submodule of \( QB_1 \oplus \cdots \oplus QB_n \) generated by \( B_1 \oplus \cdots \oplus B_n \) together with \( q^{-1}(b_1 + \cdots + b_n) \). Then \( G \) and \( H \) are indecomposable, and in particular have no non-trivial completely decomposable summands, but \( G \oplus H \) has a completely decomposable summand of rank \( n \).
PROOF: ****

DEFINITION 4.29. We define \( G^*(t) \) as the submodule generated by \( \{ g \in G \mid t(g) > t \} \) and \( G^*[t] \) as \( \bigcap_{s < t} G[s] \). (In other words, \( g \in G^*[t] \) if and only if \( \mu(g) = 0 \) whenever \( \mu: G \to A \) where \( A \) is a rank-one module with \( t(A) < t \).

PROPOSITION 4.30. (1) \( G^*(t) \subseteq G(t) \cap G[t] \).
(2) \( G(t) + G[t] \subseteq G^*[t] \)

PROOF: (1) Clearly \( G^*(t) \subseteq G(t) \). And for all \( g \) with \( t(g) > t \), if \( \varphi: G \to A \) with \( t(A) = t \), then \( \varphi(g) = 0 \) by Proposition 2.21, so that \( g \in G[t] \). Thus \( G^*(t) \subseteq G[t] \).
(2) Since by Proposition 4.12 \( s < t \Rightarrow G[s] \supseteq G[t] \), it follows that \( G[t] \subseteq G^*[t] \). And if \( g \in G(t) \) then \( g \in G[s] \) for all \( s < t \) since \( t(g) \geq t > s \) and so if \( \varphi: G \to A \) with \( t(A) = s \) then \( \varphi(g) = 0 \). Thus \( g \in G^*[t] \), and so \( G(t) \subseteq G^*[t] \). \( \Box \)

PROPOSITION 4.31. Let \( G \) be completely decomposable and \( G = \bigoplus G_t \), where for each \( t \), \( G_t \) is \( t \)-projective. If \( t \) is any type then
(1) \( G(t) = \bigoplus_{s \geq t} G_s \).
(2) \( G^*(t) = \bigoplus_{s > t} G_s \).
(3) \( G[t] = \bigoplus_{s \geq t} G_s \).
(4) \( G^*[t] = \bigoplus_{s < t} G_s \).

PROOF: Since \( G_s \) is \( t \)-projective it is \( t \)-saturated and \( t \)-bounded.
(1) \( G_s(t) = G_s \) if \( s \geq t \) and \( G_s(t) = 0 \) if \( s \not\geq t \). Thus by Proposition 4.4
\( G[t] = \bigoplus G_s[t] = \bigoplus_{s \geq t} G_s \).
(2) \( G^*_s(t) = G_s \) if \( s > t \) and \( G^*_s(t) = 0 \) otherwise. Thus \( G^*(t) = \bigoplus G^*_s(t) = \bigoplus_{s > t} G_s \).
(3) \( G_s[t] = 0 \) if \( s \leq t \) and \( G_s[t] = G_s \) otherwise. Thus \( G[t] = \bigoplus G_s[t] = \bigoplus_{s \geq t} G_s \).
(4) \( G^*_s[t] = 0 \) if and only if \( G_s[t'] = 0 \) for some \( t' < t \) and this is true if and only if \( s < t \). Otherwise \( G^*_s[t] = G_s \). Thus \( G^*[t] = \bigoplus G^*_s[t] = \bigoplus_{s < t} G_s \). (WHY?) \( \Box \)

If \( g \in G \) then to say that \( g \in G^*[t] \) is to say that \( \varphi(g) = 0 \) for every \( \varphi: G \to Q \) such that \( t(\varphi(g)) < t \). Clearly this is true if either \( g \in G[t] \) or \( t(g) = t \). We have already taken note of this above as the formula \( G(t) + G[t] \subseteq G^*[t] \) in Proposition 4.30. It seems naively plausible that \( G[t] \) and \( G(t) \) would account for all of \( G^*[t] \), i.e. that \( G^*[t] = G(t) + G[t] \). Surprisingly enough, though, this plausible equality is valid only if \( G \) is completely decomposable.

PROPOSITION 4.32. (1) A finite rank torsion free module \( G \) is completely decomposable if and only if if and only if \( G^*[t] = G(t) + G[t] \) for all \( t \in CT(G) \).
(2) \( G \) is quasi-isomorphic to a completely decomposable module if and only if \( G^*[t] = (G(t) + G[t])_s \) for all \( t \in CT(G) \).
PROOF: By Proposition 4.26 $G(t) + G[t]$ is a quasi-pure submodule of $G$, so $G^*[t] = (G(t) + G[t])_*$ if and only if $G^*[t]$ is quasi-equal to $G(t) + G[t]$. For every type $t$, define $F_t(G) = G^*[t]/(G(t) + G[t])$. (Note that $F_t(G)$ need not be torsion free.) Then $F_t$ is a functor in the obvious way, and respects direct sums since it is additive. The assertion then is that $G$ is completely decomposable $\iff F_t(G) = 0$ for all $t \in \text{CT}(G)$ and $G$ is quasi-isomorphic to a completely decomposable module if and only if $F_t(G)$ has finite length for all $t$.

$(\Rightarrow)$: Since $F_t$ respects direct sums, to see that $F_t(G) = 0$ for all types $t$ when $G$ is completely decomposable it suffices to prove that $F_t(G) = 0$ for modules $G$ with rank $G = 1$. Now if rank $G = 1$ and $t \leq t(G)$ then $G(t) = G^*[t] = G$, so $G = G(t) + G[t] = G^*[t]$ and $F_t(G) = 0$. On the other hand, if $t > t(G)$ then $G^*[t] = 0$ and again $F_t(G) = 0$. Finally, if rank $G = 1$ and $t$ and $t(G)$ are incomparable, then $G^*[t] = G^*[t] = G$ so that $G = G[t] + G(t) = G^*[t]$ and $F_t(G) = 0$ in this case as well.

Now if $G$ is quasi-equal to a completely decomposable module $G'$ then $G^*[t]$ is quasi-equal to $G^*[t]$ and $G(t) + G[t]$ is quasi-equal to $G'(t) + G'[t]$. Thus if $G^*[t] = G(t) + G[t]$ then $G^*[t]$ is quasi-equal to $G'(t) + G'[t]$.

$(\Leftarrow)$: Now suppose $F_t(G)$ has finite length for all $t \in \text{CT}(G)$. We will prove by induction on rank that $G$ is quasi-isomorphic to a completely decomposable module, and that if in fact $F_t(G) = 0$ for all $t$ then $G$ is completely decomposable. If rank $G = 1$ or, more generally, if $\text{CT}(G) = \{t\}$ for some $t$, then $G[t] = 0$ and $G^*[t] = G$, so that $F_t(G) = 0$ implies that $G = G(t)$, so $G$ is both $t$-saturated and $t$-bounded, hence is $t$-projective and a fortiori completely decomposable. Now in general note that by Proposition 2.26 $\text{CT}(G)$ must contain minimal elements. If $\text{CT}(G)$ contains more than one element, let $s$ be minimal in $\text{CT}(G)$. Then $G[s'] = 0$ for all $s' < s$, so $G^*[s] = G$. The hypothesis applied for $t = s$ together with Proposition 4.26 yields that $G = G^*[s]$ is quasi-equal to $G(s) + G[s] = G_s \oplus G[s]$ for some $s$-projective module $G_s$. Then for any $t$ the fact that $F_t$ is an additive functor yields $F_t(G) = F_t(G_s) \oplus F_t(G[s])$. Then by induction on rank we conclude that $G[s]$ is quasi-equal to a completely decomposable module, so that $G_s + G[s]$ is quasi-equal to a completely decomposable module, and thus $G$ is quasi-equal to a completely decomposable module and is completely decomposable if $F_t(G) = 0$ for all $t$.  

In Proposition 4.19 we saw that a module quasi-isomorphic to a $t$-projective module is also $t$-projective. It is not in general true, however, that a module quasi-isomorphic to a completely decomposable module is completely decomposable. For instance in Example 1.49 the module $H$ formed from the completely decomposable module $B \oplus C$ by adjoining an element $(b + c)/w$ is quasi-isomorphic to $B \oplus C$ but is not itself completely decomposable: in fact, it is indecomposable and has rank two. $H$ was obtained by filling in glue in the space between the two incomparable modules $B$ and $C$. If we had tried to add glue on top of $B$ or $C$, by adjoining to $B \oplus C$ an element $b/w$, for instance, with $b \in B$, the construction would not have worked. It would also not have worked if $t(B) = t(C)$ since in that case by Proposition 4.19 $H$ would be $t$-projective. In fact, if $t(B) = t(C)$ and $H = (B \oplus C) + W(b + c)/w$, then let $B'$ be the rank-one pure
submodule of $H$ containing $(b + c)/w$. Then $B' \sim B \sim C$ and $H = B' \oplus C$. We will also see in Chapter 5 that $H$ is necessarily completely decomposable if $t(B) \leq t(C)$ (Proposition 5.4).

Modules quasi-isomorphic to completely decomposable modules are called almost completely decomposable and will be studied in Chapter 5.

The idea that one can test for complete decomposability of a module $G$ by seeing whether a family of functors $F_t$ vanish on $G$ is quite fascinating. As an immediate consequence, we get the first part of the following classical theorem.

**Theorem 4.33.** (1) A direct summand of a completely decomposable module is completely decomposable.

(2) If $G$ is completely decomposable, and $G = \bigoplus G_t$ where the $G_t$ are $t$-projective modules, then the $G_t$ are uniquely determined up to isomorphism. In fact, $G_t \approx (G(t) + G[t])/G[t] \approx G(t)/G^*(t)$.

**Proof:** (1) For any type $t$, define the functor $F_t$ as in the proof of the preceding Proposition. Then $G$ is completely decomposable if and only if $F_t(G) = 0$ for all $t \in CT(G)$. But since $F_t$ is an additive functor, $F_t(H \oplus K) = F_t(H) \oplus F_t(K)$. Hence if $H \oplus K$ is completely decomposable, then so are $H$ and $K$.

(2) Let $G = \bigoplus G_t$ where each $G_t$ is $t$-projective. By Proposition 4.31 $G(t) = \bigoplus_{s \geq t} G_s$ and $G[t] = \bigoplus_{s \geq t} G_s \supseteq \bigoplus_{s > t} G_s$, so that $G(t) + G[t] = G_t \oplus G^*(t)$. Thus $G_t$ is the same $G_t$ as occurred in Proposition 4.32 and $G_t \approx (G(t) + G[t])/G[t]$. Furthermore since $G^*(t) = \bigoplus_{s > t} G_s$, $G_t \approx G(t)/G^*(t)$.

**Corollary 4.34.** If $W$ is a principal ideal domain and $G$ and $H$ are quasi-isomorphic completely decomposable modules, then $G$ and $H$ are in fact isomorphic.

**Proof:** If $G = \bigoplus G_t \sim H = \bigoplus H_t$, where for each $t$ the $G_t$ and $H_t$ are $t$-projective, then by Theorem 3.24 $G_t \sim H_t$ for each $t$. Thus it suffices to suppose that $G$ and $H$ are $t$-projective for some type $t$. By Corollary 2.10 if $W$ is a principal ideal domain then the rank-one summands of $G$ are isomorphic to the rank-one summands of $H$. Thus $G \approx H$.

**Warfield Isomorphism Theorems.** In the results that follow it is worth remembering that by Proposition 2.2, if $A$ is a subring of $W$ then $A = S^{-1}W$ for some multiplicative set $S$ and that for any $G$, $A \otimes G$ is naturally isomorphic to $S^{-1}G$.

**Proposition 4.35.** Let $A$ have rank one and let $A_0 = \text{End } A$. Recall that $t(A_0) = [t(A) : t(A)]$. Let $G$ and $H$ be finite rank torsion free modules.

1. $G(t) \approx A_0 \otimes G(t)$.
2. $G(t) \approx A \otimes \text{Hom}(A, G)$.
3. $\text{Hom}(A, A \otimes G) \approx A_0 \otimes G$.
4. $\text{Hom}(A \otimes G, A \otimes H) \approx \text{Hom}(G, A_0 \otimes H) \approx \text{Hom}(A_0 \otimes G, A_0 \otimes H)$. 

PROOF: (2) Already proved in Proposition 4.3.

(3) Since \( t(A \otimes G) \geq t = OT(A) \), \( A \otimes G \) dominates \( A \). Thus by Proposition 4.11 \( \text{Hom}(A, A \otimes G) \approx \text{Hom}(A, A) \otimes G = A_0 \otimes G \).

(4) By (3) and the adjointness between \( \text{Hom} \) and the tensor product, 
\[ \text{Hom}(A \otimes G, A \otimes H) \approx \text{Hom}(G, \text{Hom}(A, A \otimes H)) \approx \text{Hom}(G, A_0 \otimes H) \approx \text{Hom}(A_0 \otimes G, A_0 \otimes H) \] where the last isomorphism holds by Proposition 0.*, since \( A_0 \) is a subring of \( Q \).

PROPOSITION 4.36. If \( A \) is a subring of \( Q \) and \( t = t(A) \) then

1. \( G(t) \approx \text{Hom}(A, G) \).
2. For any \( H \), \( \text{Hom}(G, H)(t) = \text{Hom}(G, H(t)) \).
3. For any \( G \), \( \text{Hom}(G, A) \) is a \( t \)-projective module.

PROOF: (1) Since \( A \) is a subring of \( Q \) then \( \text{Hom}(A, G) \) is an \( A \) module. Furthermore \( A = S^{-1}W \) for some multiplicative set \( S \) and so by Proposition 4.7 and Proposition 0.*, 
\[ G(t) \approx A \otimes \text{Hom}(A, A) = A \otimes_A \text{Hom}(A, G) \approx \text{Hom}(A, G) \].

(2) By (1), \( \text{Hom}(G, H)(t) = \text{Hom}(A, \text{Hom}(G, H)) \approx \text{Hom}(A \otimes G, H) \approx \text{Hom}(G, \text{Hom}(A, H)) \approx \text{Hom}(G, H(t)) \).

(3) ****

COROLLARY 4.37. If \( R \) is any dedekind domain over \( W \) (not necessarily with finite rank) then for any finite rank torsion free \( W \)-module \( G \), \( \text{Hom}(G, R) \) is a projective \( R \)-module.

PROOF: By Proposition 0.*, \( \text{Hom}(G, R) \approx \text{Hom}_R(R \otimes G, R) \). But \( R \otimes G \) is a finite rank torsion free \( R \)-module. Therefore we may as well replace the ground ring by \( R \), in other words no generality is lost in supposing \( R = W \) so that the result reduces to proposition 4.36.

The following corollary is essentially just an elaboration of what was proved in Proposition 4.14. We take the opportunity, though, to give a different proof.

COROLLARY 4.38. Let \( A \) be a subring of \( Q \) and \( t = t(A) \). Then for any \( G \) the following conditions are equivalent:

1. \( G \) is \( t \)-saturated.
2. \( G \) is an \( A \)-submodule of \( QG \).
3. \( A \otimes G \approx G \).
4. \( \text{Hom}(A, G) \approx G \).

PROOF: (1) \( \Rightarrow \) (4): By Corollary 4.37.

(4) \( \Rightarrow \) (3): As noted in the proof of Corollary 4.37, \( A \otimes \text{Hom}(A, G) \approx \text{Hom}(A, G) \).

(3) \( \Rightarrow \) (2): Clear since \( A \otimes G \) is an \( A \)-submodule of \( QA \otimes G \).

(2) \( \Rightarrow \) (1): If \( G \) is an \( A \)-module then by Proposition 0.* \( \text{Hom}(A, G) \approx \text{Hom}_A(A, G) \approx G \). Thus by Corollary 4.37 \( G(t) \approx G \). Thus \( \text{rank} G(t) = \text{rank} G \). Since \( G(t) \) is a pure submodule of \( G \), we see that \( G(t) = G \), i.e. \( G \) is \( t \)-saturated.
Corollary 4.39. Let $A$ be a rank-one module and let $G$ and $H$ be $t(A)$-saturated modules. Then

1. The map $\text{Hom}(G, H) \to \text{Hom}(A \otimes G, A \otimes H)$ given by $\varphi \mapsto 1_A \otimes \varphi$ is an isomorphism.

2. The map $\text{Hom}(G, H) \to \text{Hom}(\text{Hom}(A, G), \text{Hom}(A, H))$ given by $\varphi \mapsto \varphi_*$ is an isomorphism.

Proof: (1) If $A_0 = [A : A]$ then by Corollary 4.38 and Proposition 4.35 $\text{Hom}(G, H) \approx \text{Hom}(A_0 \otimes G, A_0 \otimes H) \approx \text{Hom}(A \otimes G, A \otimes H)$.

(2) Since by Proposition 4.35 the natural maps $\sigma_G : A \otimes \text{Hom}(A, G) \to A$ and $\sigma_H : A \otimes \text{Hom}(A, H) \to H$ are isomorphisms, it follows from (1) that

$$\text{Hom}(\text{Hom}(A, G), \text{Hom}(A, H)) \approx \text{Hom}(A \otimes \text{Hom}(A, G), A \otimes \text{Hom}(A, H)) \approx \text{Hom}(G, H).$$

Let $\eta$ denote the composition of these isomorphisms and let $\theta : \text{Hom}(G, H) \to \text{Hom}(\text{Hom}(A, G), \text{Hom}(A, H))$ be given by $\theta(\varphi) = \varphi_*$. Now let $\varphi \in \text{Hom}(G, H)$ and $g \in G$. Then $\eta\theta(\varphi)(g) = \eta(\varphi_*)(g) = \sigma_H(\varphi_*\sigma_G^{-1}(g)) = \sigma_H\sigma_G^{-1}(\varphi(g)) = \varphi(g)$ by the naturality of $\sigma$. Thus $\eta\theta(\varphi) = \varphi$ so that $\eta\theta = 1$.

On the other hand, let $\psi \in \text{Hom}(\text{Hom}(A, G), \text{Hom}(A, H))$ and $\mu \in \text{Hom}(A, G)$. Then

$$\theta\eta(\psi)(\mu) = \theta(\sigma_H(1 \otimes \psi)\sigma_G^{-1})(\mu)$$
$$= (\sigma_H(1 \otimes \psi)\sigma_G^{-1})_*(\mu)$$
$$= \sigma_H(1 \otimes \psi)\sigma_G^{-1}\mu$$
$$= (1 \otimes \psi)\sigma_G\sigma_G^{-1}\mu$$
$$= \psi(\mu)$$

so that $\theta\eta(\psi) = \psi$. Thus $\eta$ is an isomorphism with inverse $\theta$. \(\Box\)

Corollary 4.40. If $[t(A) : t(A)] \leq \text{IT}(G)$ then $\text{Hom}(A, A \otimes G) \approx G$ and $\text{Hom}(A \otimes G, A \otimes H) \approx \text{Hom}(G, H)$.

Proof: If $A_0 = \text{End} A$ then by hypothesis $G$ and $H$ are $t(A_0)$-saturated. Thus by Proposition 4.* $A_0 \otimes G \approx G$ and $A_0 \otimes H \approx H$, so the result follows from Proposition 4.35. \(\Box\)

Corollary 4.41. Let $A$ be a subring of $Q$ and $t \geq t(A)$. Then for any $G$,

1. $A \otimes G[t] = (A \otimes G)[t]$.
2. $A \otimes G[t]$ is a summand of $A \otimes G$.
3. $A \otimes (G/G[t])$ is $t$-projective.
PROOF: (1) We can identify $G$ with the set of elements $1 \otimes g \in A \otimes G$. Thus if $x \in G[t]$ then $1 \otimes x \in (W \otimes G)[t]$ and if $\varphi: A \otimes G \to A$ then since by Proposition 0.* $\varphi$ is $A$-linear, $\varphi(a \otimes x) = a \varphi(1 \otimes x) = 0$. Thus $A \otimes G[t] \subseteq (A \otimes G)[t]$ without any assumptions on $t$. Therefore by Proposition 4.12 it suffices to prove that 

$$(A \otimes G)/(A \otimes G[t])$$

is $t$-bounded. In fact, $(A \otimes G)/(A \otimes G[t]) \cong A \otimes G/G[t]$ and 

$$\text{OT}(A \otimes G/G[t]) = \text{OT}(A)\text{OT}(G/G[t]) = t(A)\text{OT}(G/G[t]) \leq (t(A))^2 = t(A) \leq t$$

since $t(A)$ is idempotent by Proposition 2.*.

(3) By (1), $A \otimes (G/G[t]) \cong (A \otimes G)/(A \otimes G[t])$, and this is $t$-bounded. By Corollary 4.40 $A \otimes (G/G[t])$ is also $t$-saturated. Thus by Proposition 4.17 it is $t$-projective.

(2) By Corollary 4.40 $A \otimes G$ is $t$-saturated. Since $A \otimes (G/G[t])$ is $t$-projective by (3), $A \otimes G[t]$ is a summand of $A \otimes G$ by Baer's Lemma (Theorem 4.16). \checkmark

PROPOSITION 4.42. (1) If $K$ is a $t$-bounded module, then for any $G$, $\text{Hom}(G, K)$ is $t$-bounded.

(2) If $t$ is idempotent and $K$ is $t$-saturated, then $\text{Hom}(G, K)$ is $t$-saturated.

PROOF: (1) Let $K$ be $t$-bounded. Let $\delta: G \to \text{Hom}(\text{Hom}(G, K), K)$ be defined by $\delta(g)(\varphi) = \varphi(g)$. Then by Proposition 4.4, for any $g \in G$, $\delta(g)$ maps $\text{Hom}(G, K)[t]$ into $K[t] = 0$. Thus if $\varphi \in \text{Hom}(G, K)[t]$ and $g \in G$ then $\varphi(g) = \delta(g)(\varphi) = 0$. Since this is true for all $g \in G$, thus $\varphi = 0$. Hence $\text{Hom}(G, K)[t] = 0$ so $\text{Hom}(G, K)$ is $t$-bounded.

(2) If $t$ is idempotent then by Proposition 2.15 $t = t(A)$ for some subring $A$ of $Q$. By Proposition 4.14 if $K$ is saturated then $K$ is an $A$-module. But then $\text{Hom}(G, K)$ is also an $A$-module, hence is also $t$-saturated. \checkmark

PROPOSITION 4.43. Let $A$ be a rank-one module and $t = t(A)$. Then for any $G$, $\text{IT}(\text{Hom}(A, G)) = [\text{IT}(G(t)): t]$.

PROOF: First note that $\text{Hom}(A, G) = \text{Hom}(A, G(t))$. If $B$ is a pure rank-one submodule of $G(t)$, then by Proposition 1.37 $\text{Hom}(A, B)$ is a pure rank-one submodule of $\text{Hom}(A, G)$. Furthermore these are all the pure rank-one submodules of $\text{Hom}(A, G)$ since if $\varphi \in \text{Hom}(A, G)$ then $\varphi \in \text{Hom}(A, \varphi(A)_+ \varphi(A)_+)$. But by Proposition 2.*, $t(\text{Hom}(A, B)) = [t(B): t]$. Taking the greatest lower bound over all $B \lhd G(t)$ yields $\text{IT}(\text{Hom}(A, G)) = [\text{IT}(G(t)): t]$. \checkmark

COROLLARY 4.44. For any $W$-module $G$, there exists a rank-one module $A$ with locally trivial type and a module $G_0$ with $\text{IT}(G_0) = [\text{IT}(G): \text{IT}(G)]$ (so that, in particular, $\text{IT}(G_0)$ is idempotent) such that $G \approx A \otimes G_0$.

PROOF: Let $t' = \text{IT}(G)$, let $t_0 = [t': t']$, and let $A'$ be a rank-one module with $t' = t(A')$. Let $G_0 = \text{Hom}(A', G)$. By Proposition 4.43, $\text{IT}(G_0) = [\text{IT}(G(t)): t'] = [\text{IT}(G): t'] = [t': t'] = t_0$. By Proposition 4.35, $G = G(t') \approx A' \otimes \text{Hom}(A', G) = A' \otimes G_0$. Now by Proposition 2.19 $t'$ can be written $t' = t[t': t'] = tt_0$, where $t$ is locally trivial. Thus if $A$ is a rank-one module with $t(A) = t$ and if $A_0 = [A': A']$ (so that $t(A_0) = \text{IT}(G_0)$) then $G \approx A' \otimes G_0 \approx A \otimes A_0 \otimes G_0 \approx A \otimes G_0$, since $A \otimes A_0 \approx A_0 A = A$. (WHY?) \checkmark
We define the $t$-rank of $G$ as $\text{rank} \text{Hom}(G, A)$ where $A$ is a rank-one module with $t(A) = t$.

**Proposition 4.45.** Let $A$ be a rank-one module with type $t$.

1. $t$-rank $G = \text{rank} G/G[t]$.
2. $\text{Hom}(G, A)$ is $t$-bounded.
3. If $t$ is idempotent, then $\text{Hom}(G, A)$ is $t$-projective.
4. If $t$ is idempotent then $t$-rank $G$ is the rank of a maximal $t$-projective quasi-direct summand of $A \otimes G$.
5. If $t$ is idempotent then $t$-rank $(G \otimes H) = (t$-rank $G)(t$-rank $H)$.

**Proof:**

1. This was proved in Proposition 4.5.
2. This is a special case of Proposition 4.42.
3. If $t$ is idempotent, then by Proposition 4.42 $\text{Hom}(G, A)$ is $t$-saturated. Since it is $t$-bounded by (2), by Proposition 4.17 it is $t$-projective.
4. If $t$ is idempotent then $\text{Hom}(G, A) \approx \text{Hom}_A(A \otimes G, A)$ so that $t$-rank $G = t$-rank $(A \otimes G)$ and in we lose no generality in thinking of $A$ as the ground ring. In other words, wlog we may suppose $A = W$. But then $t$-rank $G = \text{rank} \text{Hom}(G, W) = \text{rank} G/G[t]$. But by Proposition 4.26 $G/G[t] = G(t)/G[t]$ is isomorphic to a maximal $t$-projective summand of $G$.
5. $\text{Hom}(G \otimes H, A) \approx \text{Hom}(G, \text{Hom}(H, A))$. Now $\text{Hom}(H, A)$ is an $A$-module, hence $t$-projective by (2), i.e. is a direct sum of $s$ rank-one modules with type $t$, where $s = t$-rank $H$. Thus $t$-rank $(G \otimes H) = \text{rank} \text{Hom}(G \otimes H, A) = \text{rank} \text{Hom}(G, \text{Hom}(H, A)) = (t$-rank $H)(\text{rank} \text{Hom}(G, A)) = (t$-rank $G)(t$-rank $H)$.

**Proposition 4.46.** Let $t = \text{IT}(G)$. Then

1. $G = G_t \oplus G[t]$, for some $t$-projective module $G_t$.
2. $t$-rank $G$ is the rank of a maximal $t$-projective quasi-summand of $G$. And, in fact, $G$ has a $t$-projective actual summand of this rank.

**Proof:**

1. Since $t = \text{IT}(G)$, $G = G(t) = G(t) + G[t]$. Thus by Proposition 4.26 $G = G_t \oplus G[t]$, where $G_t$ is $t$-projective.
2. By Proposition 4.26 $G_t$ is a maximal $t$-projective quasi-summand of $G$. Since by Proposition 4.45 $t$-rank $G = \text{rank} G/G[t] = \text{rank} G_t$, the result follows. And by (1), $G$ actually has a $t$-projective summand of this rank.