6. SPLITTING FIELDS AND SPLITTING RINGS

E. L. Lady

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In Chapter 5 it was seen that Butler modules are obtained by gluing together rank-one modules, and that the key to the structure of a Butler module G lies in consideration of the types $\mathbf{t}(g)$ as g ranges over QG. Using this basic idea, it is often fairly easy to look at a specific Butler module and see what's going on.

As noted at the beginning of Chapter 5, the Pontryagin module stands in sharp contrast to Butler modules, since all elements have the same type, namely $\mathbf{t}(W)$. And yet the structure of the Pontryagin module seems far from trivial. In this Chapter we will investigate the structure of modules which are like the Pontryagin module.

THE PONTRYAGIN MODULE. Recall again the construction of the Pontryagin module.

Example 1.47. Start with $W \oplus W \oplus W$ and let $\{v_i\}_1^\infty$, $\{w_i\}_1^\infty$ be sequences of elements from W such that $v_{n+1} \equiv v_n \pmod{p^n}$ and $w_{n+1} \equiv w_n \pmod{p^n}$ for all n. Assume further that there do *not* exist s, t, $u \in W$, not all trivial, such that $s + tv_n + uw_n \equiv 0 \pmod{p^n}$ for all n. To obtain G, adjoin to $W \oplus W \oplus W$ all the elements of $p^{-n}(v_n, 1, 0)$ and $p^{-n}(w_n, 0, 1)$ for $n = 1, 2, \ldots$ Then G is indecomposable and $\mathbf{t}(g) = \mathbf{t}(W)$ for all $g \neq 0 \in G$.

If we had been constructing a Butler module, we could have started with the free module $W \oplus W \oplus W$ and then added an infinite sequence of denominators under some finite set of elements, for instance by adjoining $p^{-n}(v, 1, 0)$ and $p^{-n}(w, 0, 1)$ for some fixed v, $w \in W$ and all n. In this way, the construction of Butler modules can be thought of as in some sense vertical. In contrast, the construction of the Pontryagin module seems more horizontal, since as one adds larger denominators the numerators keep sliding over, with the result that one winds up without any particular element actually being p-divisible.

To simplify matters, it will be convenient for a while to think of W as a local ring. Then the conditions $v_{n+1} \equiv v_n \pmod{p^n}$ and $w_{n+1} \equiv w_n \pmod{p^n}$ for all n are equivalent to asserting that the sequences $\{v_n\}_1^\infty$ and $\{w_n\}_1^\infty$ converge to elements in $\hat{W}$. Let

$$v = \lim_{n\to\infty} v_n \in \hat{W}$$

$$w = \lim_{n\to\infty} w_n \in \hat{W}.$$

Then the condition that there do not exist s, t, and u, not all trivial, with $s + tv_n + uw_n \equiv 0 \pmod{p^n}$ for all n amounts to the hypothesis that 1, v, and w are linearly independent over Q. From this, one sees immediately that the Pontryagin construction is not possible if $W = \hat{W}$. (This is, of course, also clear since by Proposition 1.* all $\hat{W}$-modules are completely decomposable.)

If, indeed, the ring of scalars were $\hat{W}$ rather than W, then it can be seen that G would contain $p^{-n}(v, 1, 0)$ and $p^{-n}(w, 0, 1)$ for all n and these sequences would generate a p-divisible submodule of G with rank 2. In fact, let G^* denote the $\hat{W}$-module constructed according to the same pattern as the Pontryagin module, i.e. G^* is generated by $\hat{W} \oplus \hat{W} \oplus \hat{W}$ together with $p^{-n}(v_n, 1, 0)$ and $p^{-n}(w_n, 0, 1)$ for $n = 1, 2, \ldots$. Then for each n,

$$p^{-n}(v, 1, 0) \subseteq p^{-n}(v_n, 1, 0) + p^{-n}(v - v_n, 1, 0) \subseteq G^*$$

since $v - v_n \in p^n \hat{W}$, so that $p^{-n}(v - v_n, 1, 0) \subseteq \hat{W} \oplus \hat{W} \oplus \hat{W} \subseteq G^*$. Analogous reasoning shows that $p^{-n}(w, 0, 1) \subseteq G^*$ for all n.

This observation is in fact the key to understanding the sort of “horizontal” or “sliding” construction used in constructing the Pontryagin module. The $\hat{W}$-module G^* constructed in the previous paragraph turns out to be generated simply by $\hat{W} \oplus \hat{W} \oplus \hat{W}$ together with $p^{-n}(v, 1, 0)$ and $p^{-n}(w, 0, 1)$ for all n. (Thus G^* is a completely decomposable $\hat{W}$-module.) The original Pontryagin module G then consists of those elements of G^* which have “rational” coordinates, which is to say coordinates in Q. In other words, if we think of $Q \oplus Q \oplus Q \subseteq Q\hat{W} \oplus Q\hat{W} \oplus \hat{W}$, then $G = (Q \oplus Q \oplus Q) \cap G^*$.

This example is fundamental.

THE KUROSH THEOREM. Obviously the above is a trick that can be used to construct a large number of modules over a local ring W, enabling us to see them as the “rational part” of a completely decomposable $\hat{W}$-module. In this chapter we will show that *every* finite rank torsion free module over a discrete valuation ring W in fact can be constructed in this way.

Furthermore, we will see that the idea of this construction can be used in many situations for finite rank torsion free modules over rings that are not local.

In order to see that all finite rank modules over a discrete valuation ring can be constructed as above, i.e. as the intersection of a vector space over Q and a completely decomposable $\hat{W}$-module, there are four points to establish:

BASIC STEPS 6.1. (1) If G is a finite rank torsion free module over a discrete valuation ring W, then there exists a $\hat{W}$-submodule G^* of $\hat{W} \otimes QG$ such that $G = QG \cap G^*$. In fact, one can choose $G^* = \hat{W} \otimes G$.

(2) The $\hat{W}$-module G^* has a very simple structure.

(3) Given two modules constructed this way, $G = QG \cap G^*$ and $H = QH \cap H^*$, then $\operatorname{Hom}(G, H)$ can be identified with the set of $\hat{W}$-linear maps $G^* \to H^*$ which map QG into QH.

(4) If V is a finite dimensional Q-space and G^* is an essential $\hat{W}$-submodule of $\hat{W} \otimes V$ and $G = V \cap G^*$, then G uniquely determines G^*. In fact, $G^* = \hat{W} \otimes G$.

In order to extend the scope of our results beyond the local case, we need to notice what precisely it is about $\hat{W}$ that makes the four items above hold. In fact, we will see that

Item (1) and (3) hold because W is a pure subring of $\hat{W}$.

Item (2) holds because $\hat{W}$ is a complete discrete valuation ring.

Item (4) holds because $\text{p-rank}\,\hat{W} = 1$ (which simply means that W is p-adically dense in $\hat{W}$.)

Most of these assertions do not really require the assumption that W is local. Anticipating the generalizations we will be making shortly, we prove the relevant facts in an appropriately general setting.

Recall that a ring R is called a **Murley ring** if R is reduced and $\text{p-rank}\,R \leq 1$ for all primes p.

LEMMA 6.2. *Let W be a dedekind domain and R a torsion free W-algebra with $\text{p-rank}\,R = 1$ for all primes p. Let L be an R-module and G a W-submodule of R. Then*

1. *W is a pure submodule of R.*
2. *R is a dedekind domain and the prime ideals of R are those of the form pR, where $p \in \operatorname{Spec} W$.*
3. *For any prime p, $(pR)\text{-rank}_R L = \text{p-rank}\,L$.*
4. *If L is reduced then the multiplication map $R \otimes G \to RG$ induces an isomorphism $R \star G \approx RG$, where RG denotes the R-submodule of L generated by G and, as usual, $R \star G = (R \otimes G)/\mathbf{d}(R \otimes G)$.*
5. *RG/G is divisible.*
6. *For every $w \neq 0 \in W$, $RG = G + wRG$.*

PROOF: (1) By Proposition 1.15 the fact that R is a ring and not p-divisible implies that W is p-pure in R for all primes p.

(2) & (3) R is a Murley ring, so this is Proposition 3.42.

(4) If L is reduced then by Proposition 3.42 there is an isomorphism $\zeta\colon R \star L \to L$ with $\zeta(r \star \ell) = r\ell$. Clearly $\zeta(R \star G) = RG$, so $RG \approx R \star G$.

(5) Since $RG \approx R \star G$, $\text{p-rank}\,RG = \text{p-rank}(R \star G) = (\text{p-rank}\,R)(\text{p-rank}\,G) = \text{p-rank}\,G$ for all primes p. Thus $\text{p-rank}(RG/G) = 0$ for all p so RG/G is divisible.

(6) Let $w \neq 0 \in W$. By (5) RG/G is divisible, so $RG/G = w(RG/G) = (wRG{+}G)/G$. Since $G \subseteq RG$ and $G \subseteq wRG + G$, it follows that $RG = wRG + G$. $\boxed{\checkmark}$

Item (1) of Basic Steps 6.1 is very simple, so we will dispose of it first. It is in fact a general property of faithfully flat ring extensions. At this point, we momentarily drop our earlier assumption that W is local.

PROPOSITION 6.3. *Let W be a dedekind domain and R a torsion free W-algebra such that W is pure in R. Let G be a torsion free W-module and identify QG as a subspace of $R \otimes QG$ in the usual way. Then $G = QG \cap (R \otimes G)$.*

PROOF: What is claimed is simply that G is a pure submodule of $R \otimes G$, or, more pedantically, that $W \otimes G$ is pure in $R \otimes G$. This follows from the fact that W is pure in R since by Proposition 1.38 tensoring preserves purity. $\boxed{\checkmark}$

One can note that in this proposition one did not need to assume that R is an integral domain or even commutative. In fact the very assumption that R is a ring was not needed, although that will be essential for the applications.

COROLLARY 6.4. *Let W be a pure subring of a dedekind domain R. Then* $\mathrm{Hom}(G,H)$ *consists of those* $\varphi \in \mathrm{Hom}(QG,QH)$ *such that* $1_R \otimes \varphi \in \mathrm{Hom}_R(R \otimes G,\ R \otimes H)$ *and* $\mathrm{QHom}(G,H)$ *consists of those* $\varphi \in \mathrm{Hom}(QG,QH)$ *such that* $1_R \otimes \varphi \in \mathrm{QHom}_R(R \otimes G,\ R \otimes H)$.

PROOF: (1) Clearly if $\varphi \in \mathrm{Hom}(G,H)$ then $1_R \otimes \varphi \in \mathrm{Hom}_R(R \otimes G,\ R \otimes H)$. Conversely, suppose that $\varphi\colon QG \to QH$ and $1 \otimes \varphi \in \mathrm{Hom}_R(R \otimes G,\ R \otimes H)$. Then, using the fact that $Q \cap R = W$ since $W \lhd R$, one gets $(1 \otimes \varphi)(W \otimes G) \subseteq (Q \otimes H) \cap (R \otimes H) = W \otimes H$, i.e. $\varphi(G) \subseteq H$. Thus $\varphi \in \mathrm{Hom}(G,H)$.

(2) By $\mathrm{QHom}_R(R \otimes G,\ R \otimes H)$ is simply meant the divisible hull of $\mathrm{Hom}_R(R \otimes G,\ R \otimes H)$ *as a W-module*. This consists of those $\gamma\colon R \otimes QG \to R \otimes QH$ such that $w\gamma(R \otimes G) \subseteq R \otimes H$ for some $w \neq 0 \in W$. Thus if $\varphi \in \mathrm{Hom}(QG,QH)$ then $1_R \otimes \varphi \in \mathrm{QHom}_R(R \otimes G,\ R \otimes H)$ if and only if for some $w \neq 0 \in W$, $1 \otimes w\varphi \in \mathrm{Hom}_R(R \otimes G,\ R \otimes H)$, and by (1) this is true if and only if $w\varphi \in \mathrm{Hom}(G,H)$, which is equivalent to the assertion that $\varphi \in \mathrm{QHom}(G,H)$. $\boxed{\checkmark}$

PROPOSITION 6.5. *Let R be a ring containing W as a pure subring, let G be a W-module such that $R \star G$ is a finitely generated R-module, and let H be any torsion free W-module. Identify* $\mathrm{Hom}(G,QH)$ *and* $\mathrm{Hom}_R(R \otimes G,\ R \otimes H)$ *as submodules of* $\mathrm{Hom}_R(QR \otimes G,\ QR \otimes H)$ *in the usual way. Let*

$$\mu\colon \mathrm{Hom}(QG,QH) \to \mathrm{Hom}_R(\mathbf{d}(R \otimes G),\ QR \otimes H)$$

be the map such that for $\varphi \in \mathrm{Hom}(QG,QH)$, $\mu(\varphi)$ *is the restriction of* $R \otimes \varphi$ *to* $\mathbf{d}(R \otimes G)$. *Make the obvious identification* $\mathrm{Hom}_R(\mathbf{d}(R \otimes G),\ \mathbf{d}(R \otimes H)) \subseteq \mathrm{Hom}_R(\mathbf{d}(R \otimes G),\ QR \otimes H)$. *Then*

$$\mathrm{QHom}(G,H) = \mu^{-1}(\,\mathrm{Hom}_R(\mathbf{d}(R \otimes G),\ \mathbf{d}(R \otimes H))\,).$$

PROOF: If $\varphi \in \mathrm{Hom}(G,QH)$ then let $\psi = R \otimes \varphi$, the extension of φ to a R-linear map $QR \otimes G \to QR \otimes H$. By assumption, $R \otimes G = F \oplus \mathbf{d}(R \otimes G)$ where $F \approx R \star G$ is a finitely generated projective R-module. Now by Corollary 6.4, Lemma 3.13, and Proposition 3.*

$$\begin{aligned}
\varphi \in \mathrm{QHom}(G,H) &\iff \psi \in \mathrm{QHom}_R(R \otimes G,\ R \otimes H) \\
&\iff \psi \in \mathrm{QHom}_R(F,\ R \otimes H)\ \&\ \psi \in \mathrm{QHom}_R(\mathbf{d}(R \otimes G),\ R \otimes H) \\
&\iff \psi \in \mathrm{QHom}_R(\mathbf{d}(R \otimes G),\ R \otimes H) \\
&\iff \psi(\mathbf{d}(R \otimes G)) \subseteq \mathbf{d}(R \otimes H)
\end{aligned}$$

Since $\mu(\varphi)$ is the restriction of ψ to $\mathbf{d}(R \otimes G)$, this says that $\mathrm{QHom}(G,H) = \mu^{-1}(\,\mathrm{Hom}_R(\mathbf{d}(R \otimes G),\ \mathbf{d}(R \otimes H))\,)$. $\boxed{\checkmark}$

✠ We can summarize Proposition 6.5 by saying that the following diagram is a pull-back:

$$\begin{tikzcd}
\mathrm{QHom}(G,H) \arrow[r] \arrow[d, "\subseteq"] & \mathrm{Hom}_R(\mathbf{d}(R\otimes G),\, \mathbf{d}(R\otimes H)) \arrow[d] \\
\mathrm{Hom}(QG,QH) \arrow[r] & \mathrm{Hom}_R(QR\otimes G,\, QR\otimes H).
\end{tikzcd}$$

(The upper horizontal map here is simply the restriction of the lower.) **(This is not quite right.)**

Proposition 6.3 and Corollary 6.5 are only useful if the R-module $R\otimes G$ has considerably simpler structure than the original module G. Such will be the case if W is local and $R=\hat{W}$, since in that case by Proposition 1.49 $R\otimes G$ will be the direct sum of a free R-module and a divisible module. Thus we get a theorem foreshadowed by Kurosh and explicitly due to Beaumont & Pierce [Torsion Free Rings].

THEOREM 6.6. *Let W be a discrete valuation ring and let $\hat{W}$ be its completion. Let G be a finite rank torsion free W-module and identify G as a submodule of $\hat{W}\otimes G$ in the usual way. Then*

1. *$\hat{W}\otimes G = \mathbf{d}(\hat{W}\otimes G)\oplus F$, where F is a free $\hat{W}$-module, $F\approx \hat{W}\star G$, and $\mathrm{rank}_{\hat{W}} F = \text{p-rank}\, G$.*
2. *$G = QG\cap(\hat{W}\otimes G)$.*
3. *G is determined up to quasi-equality by QG and $\mathbf{d}(\hat{W}\otimes G)$.*
4. *If H is any torsion free W-module then $\mathrm{Hom}(G,H)$ consists of those maps $QG\to QH$ such that $(\hat{W}\otimes\varphi)(\mathbf{d}(\hat{W}\otimes G))\subseteq \mathbf{d}(\hat{W}\otimes H)$.*

PROOF: (1) Of course $\mathbf{d}(\hat{W}\otimes G)$ is a summand of $\hat{W}\otimes G$ and by Proposition 1.49 $(\hat{W}\otimes G)/\mathbf{d}(\hat{W}\otimes G)$ is a free $\hat{W}$-module. Then by Proposition 6.3

$$\begin{aligned}
\mathrm{rank}_{\hat{W}}(\hat{W}\otimes G/\mathbf{d}(\hat{W}\otimes G)) &= \text{p-rank}_{\hat{W}}(\hat{W}\otimes G/\mathbf{d}(\hat{W}\otimes G))\\
&= \text{p-rank}_{\hat{W}}(\hat{W}\otimes G) = \text{p-rank}\, G.
\end{aligned}$$

(2) Since $W \lhd \hat{W}$, this follows from Proposition 6.3.

(4) By Proposition 6.5.

(3) Two W-modules G and G' are quasi-equal if and only if $QG=QG'$ and 1_{QG} belongs to both $\mathrm{QHom}(G,G')$ and $\mathrm{QHom}(G',G)$. By (4) this holds if and only if $QG=QG'$ and $\mathbf{d}(\hat{W}\otimes G)=\mathbf{d}(\hat{W}\otimes G')$. $\boxed{\checkmark}$

Theorem 6.6 says that over a discrete valuation ring W we can identify a finite rank torsion free module G, up to quasi-equality, with a finite dimensional vector space QG and a $Q\hat{W}$-subspace D of $\hat{W}\otimes QG$ and that this yields a full embedding of the category of finite rank torsion free modules into a category where the objects consist of pairs consisting of a Q-vector space V and a $\hat{W}$-subspace of $\hat{W}\otimes V$. To get a category equivalence, the piece that is still missing is point (4) in the Basic Steps above, namely that any such pair (V,D) arises in this manner from some G.

Now given V and D, it is in fact clear from the preceding how the desired module G should be constructed. Namely we choose a free $\hat{W}$-submodule F of $\hat{W} \otimes V$ such that $F \cap D = 0$ and $F \oplus D$ is an essential $\hat{W}$-submodule of $\hat{W} \otimes V$. Then G should be $V \cap (F \oplus D)$. It is clear that $QG = V$ and it remains to be shown that $\mathbf{d}(\hat{W} \otimes G) = D$. Once again, the relevant proposition does not require the assumption that W be local.

LEMMA 6.7. *Let W be a dedekind domain and R a ring such that* $\text{p-rank}\, R = 1$ *for all primes p. Let L be a R-module which is torsion free as a W-module and G a pure W-submodule of L such that RG is an essential W-submodule of L, where RG is the R-submodule of L generated by G. Then $L = RG$.*

PROOF: Let $\ell \in L$. Since RG is an essential W-submodule of L, there exists $0 \neq w \in W$ with $w\ell \in RG$. But Lemma 6.2 $RG = G + wRG$ and so $w\ell \in G + wRG$, say $w\ell = g + wx$ for $x \in RG$. Then $g = w(\ell - x) \in G \cap wL = wG$ because $G \lhd L$. Say $g = wg_1 = w(\ell - x)$ for $g_1 \in G$. Since L is torsion free, $g_1 = \ell - x$, so $\ell = g_1 + x \in G + RG = RG$. Thus $L \subseteq RG$ and since the opposite inclusion is trivial the result is established. ☑

LEMMA 6.8. *Let W be a dedekind domain and let R be a containing W as a pure subring and such that* $\text{p-rank}\, R = 1$ *for every prime p. Let V be a Q-vector space and D a QR-subspace of $R \otimes V$. Let $G = V \cap (D \oplus F)$, where F is a free R-module such that $D \oplus QF = R \otimes V$. Then $QG = V$ and $\mathbf{d}(R \otimes G) = D$.*

PROOF: The intersection $V \cap (D \oplus F)$ is meaningful since we identify V with $W \otimes V \subseteq R \otimes V$. More precisely, if $\beta \colon V \to R \otimes V$ is the canonical monomorphism, then the construction defines $G = \beta^{-1}(D \oplus F)$. Now if we let L in Lemma 6.7 be $D \oplus F$, then $G \lhd L$ since $G = L \cap V$ and $R \otimes G \subseteq D \oplus F$ since $W \otimes G \subseteq D \oplus F$ and $D \oplus F$ is closed under multiplication by R. Moreover $QG = V \cap Q(F \oplus D) = V \cap (R \otimes V) = V$, and so $Q(R \otimes G) = R \otimes V$ and thus $R \otimes G$ is an essential submodule of $R \otimes V$ and hence a fortiori an essential submodule of $D \oplus F$. Thus by Proposition 6.7, $D \oplus F = RG$, which really means that $D \oplus F = R(W \otimes G) = R \otimes G$ so that $\mathbf{d}(R \otimes G) = \mathbf{d}(D \oplus F) = D$. ☑

This shows that the construction described in the remarks following Theorem 6.6 works. The following theorem summarizes everything so far. (This theorem will be generalized below as Theorem 6.**.)

THEOREM 6.9. *Let W be a discrete valuation ring and $\hat{W}$ its completion. Consider the category where the objects consist of pairs (V, D), with V a finite dimensional Q-space and D a $Q\hat{W}$-subspace of $Q\hat{W} \otimes V$, and where the morphisms from (V, D) to (V', D') consist of the Q-linear maps $\varphi \colon V \to V'$ such that $(\hat{W} \otimes \varphi)(D) \subseteq D'$. Then this category is isomorphic to the category of finite rank torsion free W-modules under quasi-homomorphisms.*

PROOF: Almost everything is done. There is a functor in one direction sending G to the pair $(QG, \mathbf{d}(\hat{W} \otimes G))$ and a functor in the other direction sending a pair (V, D) to $V \cap (D \oplus F)$ where F is a free $\hat{W}$-submodule of $\hat{W} \otimes V$ such that $D \oplus F$ is essential in $\hat{W} \otimes V$. If $V = QG$ and $D = \mathbf{d}(\hat{W} \otimes G)$ then since $\mathbf{d}(D \oplus F) = D = \mathbf{d}(\hat{W} \otimes G)$, it follows from Theorem 6.6 that $D \oplus F$ is quasi-equal to $\hat{W} \otimes G$. Then Proposition 6.3 and Lemma 6.8 show that these two functors are inverse to each other. $\boxed{\checkmark}$

It is strongly recommended that the reader consult [Fuchs] for the formulation of this theorem in matrix terms. Your author's personal opinion, however, is that the matrix formulation is a red herring and has been, in fact, what has stood in the way of the Kurosh Theorem receiving the appreciation to which it was justly due.

A REVERSE PERSPECTIVE. We have seen that the pair $(QG, \mathbf{d}(\hat{W} \otimes G))$ determines a quasi-equality class of modules G over a discrete valuation ring W. To determine G precisely, rather than merely up to quasi-equality, one needs to specify a free $\hat{W}$-module F such that $\hat{W} \otimes G = \mathbf{d}(\hat{W} \otimes G) \oplus F$. Thus instead of associating to G a pair $(QG, \mathbf{d}(\hat{W} \otimes G))$, one might prefer a triple $(QG, \mathbf{d}(\hat{W} \otimes G), F)$. The problem with this more precise construction, however, is a lack of functoriality since there is no canonical choice for the $\hat{W}$-module F. In fact, F and F' will produce the same G if and only if $F \oplus \mathbf{d}(\hat{W} \otimes G) = F' \oplus \mathbf{d}(\hat{W} \otimes G)$.

There is an approach to the Kurosh Theorem which is functorial and yet does not require identifying W-modules which are quasi-equal. This involves looking at the previous construction upside down.

EXAMPLE 1.47 [CONTINUED]. Consider again the example of the Pontryagin module G, assuming again that W is local. Since $\text{p-rank}\, G = 1$, by Proposition 1.49 the completion $\hat{G}$ is isomorphic to $\hat{W}$. Since G is reduced, by Proposition 1.* G is isomorphic to a pure submodule of $\hat{G}$, and thus isomorphic to a pure submodule of $\hat{W}$. More specifically, if V is the Q-subspace of $Q\hat{W}$ generated by 1, w, and w, then $G \approx V \cap \hat{W}$. (Here, as above, $v = \lim_{n\to\infty} v_n \in \hat{W}$ and $w = \lim_{n\to\infty} w_n$.)

PROOF: Consider the map $\varphi\colon Q \oplus Q \oplus Q \to Q\hat{W}$ given by $\varphi(1,0,0) = -1 \in \hat{W}$, $\varphi(0,1,0) = v$ and $\varphi(0,0,1) = w$. Then $\varphi(v_n,1,0) = v - v_n \in p^n\hat{W}$ and $\varphi(w_n,0,1) = w - w_n \in p^n\hat{W}$ so that $\varphi(p^{-n}(v_n,1,0)) \subseteq \hat{W}$ and $\varphi(p^{-n}(w_n,0,1)) \subseteq \hat{W}$ for all n, so that $\varphi(G) \subseteq \hat{W}$.

Since $1 \in \varphi(G)$ and $v_n, w_n \in W$, it follows that $v, w \in \varphi(G)$. Since $\operatorname{rank}\varphi(G) = \operatorname{rank} G = 3$, if V is the Q-subspace of $Q\hat{W}$ generated by 1, v, and w, then $Q\varphi(G) = V$. We now claim that $\varphi(G) \lhd \hat{W}$. Since $1 \in \varphi(G)$, $W \subseteq \varphi(G)$ and since by Proposition 1.* $W \lhd \hat{W}$ it suffices to prove that $\varphi(G)/W \lhd \hat{W}/W$. It will in fact suffice to prove that $\varphi(G)/W$ is divisible (Proposition 1.*), and since W is local it suffices to prove that $\varphi(G)/W$ is p-divisible. Now for all n, $v \equiv v - v_n \pmod{W}$ and $v - v_n \in p^n\varphi(G)$, so $v + W \in p^n(\varphi(G)/W)$, and likewise $w + W \in p^n(\varphi(G)/W)$. But the cosets $v+W$ and $w+W$ are linearly independent in $\varphi(G)/W = 2$ and $\operatorname{rank}\varphi(G)/W = 2$. Thus $\varphi(G)/W$ is p-divisible and hence divisible since by assumption W is local. By

Proposition 6.* $\varphi(G)/W \ \lhd\ \hat{W}/W$ and since by Proposition 6.* $W \ \lhd\ \hat{W}$, it follows that $\varphi(G) \ \lhd\ \hat{W}$. $\boxed{\checkmark}$

In general, if G is any finite rank torsion free module over a discrete valuation ring W and $\hat{G}$ is its completion, then by Proposition 1.* $G/\mathbf{d}(G)$ is canonically isomorphic to a pure submodule of $\hat{G}$ and by Proposition 1.* $\hat{G}$ is a free $\hat{W}$-module with rank equal to p-rank G. (It is also worth noting that by Proposition 1.* $\hat{G}$ can be canonically identified with $\hat{W}\star G$.) One can thus represent $G/\mathbf{d}(G)$ as an intersection $V\cap L$ where V is a finite dimensional Q-space and L is the finitely generated $\hat{W}$-module $\hat{G}$. Since $\mathbf{d}(G)$ is simply a Q-vector space, and is a summand of G, one loses little generality restricting attention to those G with $\mathbf{d}(G)=0$. Furthermore if $G=V\cap L$ and $G'=V'\cap L'$ then a homomorphism from G to G' will correspond to a map $\psi\colon L=\hat{G}\to\hat{G}'=L'$ such that $\psi(V)\subseteq V'$. (Here we have written $\psi(V)$ in accord with our standing convention that we think of ψ as being actually defined on QL.) The parallel with the earlier construction suggests the requirement that ψ should be $\hat{W}$-linear, although on the face of it there doesn't seem to be any reason for $\hat{W}$-linearity in this case. In fact, $\hat{W}$-linearity here comes for free since $\hat{W}$ is a Murley ring and so Proposition 3.43 shows that $\operatorname{Hom}(L,L')=\operatorname{Hom}_{\hat{W}}(L,L')$.

Thus we can turn Theorem 6.9 on its head.

THEOREM 6.10. *Let W be a discrete valuation ring and $\hat{W}$ its completion. Consider the category $\mathcal{C}$ where the objects consist of pairs (L,V), with L a finitely generated $\hat{W}$-module and V a Q-subspace of QL such that $\hat{W}V=QL$ and where the morphisms from (L,V) to (L',V') consist of the $\hat{W}$-linear maps $\psi\colon L\to L'$ such that $\psi(V)\subseteq V'$. Then this category is isomorphic to the category of* reduced *finite rank torsion free W-modules under homomorphisms.*

PROOF: If G is a reduced W-module then QG generates $Q(\hat{W}\star G)$ as a $\hat{W}$-module. Thus we can associate with G the pair $(\hat{W}\star G, QG)$ in $\mathcal{C}$. Conversely, given a pair (L,V) in $\mathcal{C}$ we get a W-module $G=V\cap L$.

First, note that these correspondences are functors. In fact, if $\varphi\colon G\to H$ then φ induces maps $QG\to QH$ and $\hat{W}\star G\to\hat{W}\star H$. And if $\psi\colon (L,V)\to(L',V')$ then by definition ψ is a map $L\to L'$ such that $\psi(V)\subseteq V'$, so clearly ψ induces a map $V\cap L\to V'\cap L'$.

Now if we start with G and go up and then come back down, we get back to $QG\cap(\hat{W}\star G)=G$ because by Proposition 1.42 G is pure in $\hat{W}\star G$. Thus the composition of the two functors that way is the identity.

Now suppose we start with a pair (L,V) such that $\hat{W}V=QL$ and set $G=V\cap L$. Then $QG=V$ and $\hat{W}G$ is an essential W-submodule of L, so by Lemma 6.7 $\hat{W}G=L$. But by Lemma 6.2, $\hat{W}\star G\approx\hat{W}G$. Thus there is an isomorphism $\zeta\colon \hat{W}\star G\approx L$ which is the identity on $QG=V$. Hence ζ yields an isomorphism from the pair $(\hat{W}\star G, QG)$ to (L,V), showing that the composition of the two functors in this case is also the identity.

Thus the two functors are inverse to each other and hence give an equivalence between the two categories. $\boxed{\checkmark}$

Theorem 6.10 is really only a restatement of Theorem 6.6. In fact, suppose that $G = V \cap (D \oplus F)$, where D is a divisible $\hat{W}$-module and F a free $\hat{W}$-module and $Q(D \oplus F) = \hat{W} \otimes V$. Then $F \approx (\hat{W} \otimes G)/\mathbf{d}(\hat{W} \otimes G) = \hat{W} \star G$. And if G is reduced then $G \cap \mathbf{d}(\hat{W} \otimes G) = 0$ so the projection $\hat{W} \otimes G \to F$ induces a monomorphism $\theta \colon G \to F$. By Proposition 1.8, $\theta(G) \lhd F$, so $G \approx \theta(G) = Q\theta(G) \cap F$. Thus we have represented G as the intersection of $\hat{W} \star G$ with a Q-vector space.

The real advantage of the present formulation is that $\hat{W} \star G$ is determined functorially by G whereas the submodule F of $\hat{W} \otimes G$ and the isomorphism $\hat{W} \star G \approx F$ are not.

The following example is taken from a paper by Arnold & Dugas. It will illustrate how to apply the preceding idea to a specific calculation.

EXAMPLE 6.11. Let W be a DVR, let $\hat{W}$ be its completion, and let $u \in \hat{W}$ be such that u^3 is a unit of W. Let $i \geq 1$ and let G be the pure W-submodule of $\hat{W} \oplus \hat{W}$ generated by $(1,0)$, $(0,1)$ and $(u, p^i u^2)$. Let H be the pure submodule generated by $(1,0)$, $(0,1)$ and $(p^i u, u^2)$. Then $G \approx H$.

PROOF: There is an automorphism ψ of $\hat{W} \oplus \hat{W}$ with $\psi(1,0) = (1,1)$ and $\psi(0,1) = (-u^{-2},\, -p^i u^{-1} + p^{2i})$. (Note that $u^{-1} \in \hat{W}$ and that the matrix

$$\begin{pmatrix} 1 & 1 \\ -u^{-2} & -p^i u^{-1} + p^{2i} \end{pmatrix}$$

is an invertible $\hat{W}$-matrix.)

Now note that

$$\begin{aligned} \psi(1,0) = (1,1) &= (1,0) + (0,1) \in G \\ \psi(0,1) = (-u^{-2},\, -p^i u^{-1} + p^{2i}) &= -u^{-3}(u, p^i u^2) + p^{2i}(0,1) \in G \quad (\text{since } u^{-3} \in W\,) \\ \psi(p^i u, u^2) &= p^i u\,(1,1) + u^2(-u^{-2},\, -p^i u^{-1} + p^{2i}) \\ &= (p^i u - 1, p^{2i} u^2) \\ &= p^i(u, p^i u^2) - (1,0) \in G. \end{aligned}$$

Thus $\psi(H) \subseteq G$. Since G and $\psi(H)$ are both pure submodules of $\hat{W} \oplus \hat{W}$ with rank 3, $G = \psi(H) \approx H$. $\boxed{\checkmark}$

THE KUROSH MATRICES. Traditionally, of course, the Kurosh Matrix Theorem has indeed been presented in a matrix formulation. It is your author's opinion that the use of matrices only serves to obscure the essential simplicity of the method and has in fact been one of the things that have turned off many people to the Kurosh Theorem. However from time to time one encounters papers using the matrices, so it seems wise to indicate briefly how they work and how they relate to the construction described above.

There seem to be two versions of the matrix theorem which are generally popular. The first is given in [Fuchs § 93] and therefore may be taken to be the more standard form.

Given a module G over a discrete valuation ring W with rank n, choose a basis $g_1, \dots, g_n$ for QG. (It is preferable, although not essential, that $g_i \in G$.) Also choose a basis $d_1, \dots, d_t, f_{t+1}, \dots, f_n \in \hat{W} \otimes G$ for $Q\hat{W} \otimes G$ such that $d_1, \dots, d_t$ are a basis for for $\mathbf{d}(\hat{W} \otimes G)$ and $f_{t+1}, \dots, f_n$ are the basis for a free $\hat{W}$-module F with $\hat{W} \otimes G = \mathbf{d}(\hat{W} \otimes G) \oplus F$. For each i, $g_i \in Q\hat{W} \otimes G = \mathbf{d}(\hat{W} \otimes G) \oplus QF$, so g_i is represented in terms of the above bases by coordinates $(r_{i1}, \dots, r_{it}, s_{i,t+1}, \dots, s_{in})$ with r_{ij}, $s_{ij} \in Q\hat{W}$. (If all $g_i \in G$ then r_{ij}, $s_{ij} \in \hat{W}$.) Putting these rows together, one gets an $n \times n$ matrix

$$\mathbb{M} = \begin{pmatrix} r_{11} & \dots & r_{1t} & s_{1,t+1} & \dots & s_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ r_{n1} & \dots & r_{nt} & s_{n,t+1} & \dots & s_{nn} \end{pmatrix}$$

which is necessarily invertible.

This matrix in fact determines G as a submodule of $D \oplus F$, where D is the $Q\hat{W}$-space with basis $d_1, \dots, d_t$ and F is the free $\hat{W}$-module with basis $f_{t+1}, \dots, f_n$, and QG is the Q-subspace of $QF \oplus D$ spanned by the rows of the above matrix.

The three most immediately obvious questions are

1. When do two Kurosh matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ represent the same W-submodule of $D \oplus F$?
2. When do they represent quasi-isomorphic W-modules?
3. When do they represent isomorphic W-modules?

PROPOSITION 6.12. (1) *Two Kurosh matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ represent that same W-submodule of $Q\hat{W}^t \oplus \hat{W}^{n-t}$ if and only if there is an invertible $n \times n$ matrix $\mathbb{B}$ with entries in Q such that $\mathbb{M}_2 = \mathbb{B}\mathbb{M}_1$.*

(2) *$\mathbb{M}_1$ and $\mathbb{M}_2$ represent isomorphic W-modules if and only if there is an invertible $n \times n$ matrix $\mathbb{B}$ and an invertible $n \times n$ matrix*

$$\mathbb{C} = \begin{pmatrix} \mathbb{V} & 0 \\ \mathbb{W}_1 & \mathbb{W}_2 \end{pmatrix}$$

where $\mathbb{V}$ and $\mathbb{W}_1$ have entries in $Q\hat{W}$ and $\mathbb{W}_2$ has entries in $\hat{W}$ such that $\mathbb{M}_2 = \mathbb{B}\mathbb{M}_1\mathbb{C}$.

Restated, assertion (1) of Proposition 6.12 says that two Kurosh matrices represent the same submodule of $Q\hat{W}^t \oplus \hat{W}^k$ if and only if they can be obtained from each other by a sequence of elementary row operations **with rational coefficients.** I. e. one may interchange two rows, multiply any row by q, for $q \neq 0 \in Q$, or modify a row by adding q times another row to it, for $q \in Q$. Using any of these elementary row operations on the matrix is equivalent to multiplying it on the left by an invertible $n \times n$ matrix with coefficients in Q.

Assertion (2) says that to test for isomorphism, we may, in addition, do elementary column operations with coefficients in $Q\hat{W}$ to the first t columns of the matrices and elementary column operations with coefficients in $\hat{W}$ to the last k columns. (Here, any two of the first t columns may be interchanged with each other, and likewise for the last

k columns. Any of the first t columns may be modified by adding a multiple of any other column in the matrix to it, however in the case of the last k columns, only a multiple (with coefficient from $\hat{W}$) of one of the other columns among the last k may be added.

The criterion given in Proposition 6.12 is, to say that least, rather difficult to apply. We may recall, though, that the standard way in linear algebra of determining the equivalence of matrices is by the use of some normal form. In fact, by judicious choice of the basis $d_1, \ldots, d_s, f_{s+1}, \ldots, f_n$ one can obtain matrices in a normal form essentially given in [Arnold]. To start with, it is easy to see that one can, in fact, choose $f_{s+1}, \ldots, f_n \in G$ and then choose $g_i = f_i$ for $i = s+1, \ldots, s_n$. We then get a Kurosh matrix in the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix},$$

where here the symbol 1 represents the $k \times k$ identity matrix.

Since this matrix must be non-singular, we can further multiply on the left by an invertible $n \times n$ matrix with entries in Q to get a matrix in the form

$$\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}.$$

We will refer to this as the **Arnold normal form** of the Kurosh matrix.

✠ **[The problem with the Arnold normal form is that one can't use it globally, since the choice of the $\mathbf{g}_i$ has to depend on the prime p. This is presumably why Fuchs doesn't use it. Check whether in fact this form was used in Kurosh's original paper.]**

PROPOSITION 6.13. (1) *A submodule G of $Q\hat{W}^t \oplus \hat{W}^k$ such that G generates $Q\hat{W}^t \oplus \hat{W}^k$ as a $\hat{W}$-module is represented by a matrix $\mathbb{M}$ in Arnold normal form if and only if G contains the standard basis elements for $\hat{W}^k$. Specifically, $G = V \cap (Q\hat{W}^t \oplus \hat{W}^k)$, where V is the Q-space generated by the rows of $\mathbb{M}$.*

(2) *Two matrices in Arnold normal form $\begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & \Gamma' \\ 0 & 1 \end{pmatrix}$ represent the same submodule of $Q\hat{W}^t \oplus \hat{W}^k$ if and only if $\Gamma - \Gamma'$ has entries in Q.*

(3) *Every finite rank torsion free W-module with* p-rank *k and rank $k+t$ is isomorphic to a submodule of $Q\hat{W}^t \oplus \hat{W}^k$ represented by a matrix in Arnold normal form.*

PROPOSITION 6.14. *If G is determined, as above, by the matrix in Arnold normal form $\mathbb{M} = \begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$ then G is reduced if and only if the rows of $\begin{pmatrix} \Gamma \\ 1 \end{pmatrix}$ are linearly independent over Q.*

$G/\mathbf{d}(G)$ is isomorphic to the pure submodule of $\hat{W}^k$ generated by W^k together with the rows of Γ.

EXAMPLE 6.15. **Show how to represent Example 6.11 with Kurosh matrices.**

SPLITTING FIELDS. We continue the assumption that W is local. We have seen that Theorem 6.9 and Theorem 6.10 depend on the facts that W is a pure subring of $\hat{W}$, that p-rank $\hat{W} = 1$, and that $\hat{W}$ is a complete discrete valuation ring. However as noted in the Basic Steps 6.1, the completeness of $\hat{W}$ is important only to ensure that $\hat{W} \star G$ is a free $\hat{W}$-module. Now for that purpose no proper subring of $\hat{W}$ will do.

PROPOSITION 6.16. *If W' is a proper pure subring of $\hat{W}$ then there exists a finite rank torsion free W-module G such that $W' \star G$ is not a finitely generated W'-module.*

PROOF: If W' is a proper pure subring (i.e. W-subalgebra) of $\hat{W}$ then there exists a finite rank pure W'-submodule M of $\hat{W}$ properly containing W'. (Let $m \in \hat{W}$ with $m \notin W'$ and let M be the purification in $\hat{W}$ of the W'-submodule of $\hat{W}$ generated by W' together with m.) Furthermore p-rank W' = p-rank $\hat{W} = 1$, so W' is a Murley ring and therefore by Proposition 3.* is a dedekind domain. By Lemma 6.2, (pW')-rank$_{W'} M$ = p-rank M = p-rank $\hat{W} = 1$ and rank$_{W'} M > 1$, so M is not a projective W'-module and therefore is not finitely generated. And if G is the pure W-submodule of $\hat{W}$ generated by a set of W'-generators for M, then by 6.2 $W' \star G \approx M$, so $W' \star G$ is not a finitely generated W'-module. $\boxed{\checkmark}$

However although no proper subring of $\hat{W}$ will give us the analog of the Kuorsh Theorem for *all* finite rank torsion free W-modules G, for any specific G there is almost always a smaller pure subring R of $\hat{W}$ such that $R \star G$ is a finitely generated R-module. (Exceptions would be possible only in the case that $Q\hat{W}$ is a finitely generated field extension of Q. See Corollary 6.15 below.) This leads us to the notion of a **splitting field** for a finite rank torsion free module over a discrete valuation ring W, a concept first identified (although not under that name) in [Szekeres].

DEFINITION 6.17. Let W be a discrete valuation ring and let K be a subfield of $Q\hat{W}$ and G a W-module. Let $R = K \cap \hat{W}$. We say that K is a **splitting field** for G if $R \star G$ is a finitely generated R-module. We also say that G is **K-split** or **R-split**.

PROPOSITION 6.18. *Let W be a discrete valuation ring, let K be a subfield of $Q\hat{W}$ and let $R = K \cap \hat{W}$. Then*

1. *R is a discrete valuation ring with maximal ideal generated by the unique prime p of W.*
2. *R is a Murley ring.*
3. *If G is any W-module then* p-rank $R \otimes G$ = p-rank $R \star G$ = p-rank G.
4. *If G is K-split then $R \star G$ is a free R-module.*

PROOF: (1) & (2) Since $W \subseteq R \subseteq \hat{W}$ and $W \ \lhd \ \hat{W}$, $W \ \lhd \ R$. And since K is a Q-vector space, $R = K \cap \hat{W} \ \lhd \ \hat{W}$. Thus $1 =$ p-rank $W \leq$ p-rank $R \leq$ p-rank $\hat{W} = 1$. Therefore R is a Murley ring and so by Lemma 6.2 R is a dedekind domain with a unique prime ideal generated by p, i.e. a discrete valuation ring.

(3) By Proposition 1.37 since p-rank $R = 1$, p-rank $R \otimes G =$ p-rank $R \star G =$ (p-rank R)(p-rank G) $=$ p-rank G.

(4) If G is R-split then $R \star G$ is a finitely generated R-module. Since R is a discrete valuation ring it is a principal ideal domain and thus $R \star G$ is free over R. ☑

PROPOSITION 6.19. *Let W be a discrete valuation ring, let K be a subfield of $Q\hat{W}$, let $R = K \cap \hat{W}$, and let G be a finite rank torsion free module. Then K is a splitting field for G if and only if $\mathbf{d}(\hat{W} \otimes G)$ has a basis in $K \otimes G$.*

PROOF: We claim that $K \otimes G$ contains a basis for $\mathbf{d}(\hat{W} \otimes G)$ if and only if $\operatorname{rank}_R R \star G \leq$ p-rank G. In fact, if $K \otimes G$ contains a basis for $\mathbf{d}(\hat{W} \otimes G)$ then the K-space generated by this basis is divisible and contained in $(K \otimes G) \cap (\hat{W} \otimes G) = R \otimes G$. Hence $\operatorname{rank}_R \mathbf{d}(R \otimes G) = \dim_K \mathbf{d}(R \otimes G) \geq \operatorname{rank}_{\hat{W}} \mathbf{d}(\hat{W} \otimes G)$. On the other hand if $\dim_K \mathbf{d}(R \otimes G) \geq \operatorname{rank}_{\hat{W}} \mathbf{d}(\hat{W} \otimes G)$ then any K-basis for $\mathbf{d}(R \otimes G)$ is a $Q\hat{W}$-basis for $\mathbf{d}(\hat{W} \otimes G)$. Therefore $\mathbf{d}(\hat{W} \otimes G)$ has a basis in $K \otimes G$ if and only if $\operatorname{rank}_R \mathbf{d}(R \otimes G) \geq \operatorname{rank}_{\hat{W}} \mathbf{d}(\hat{W} \otimes G)$ or, equivalently, $\operatorname{rank}_R(R \star G) = \operatorname{rank}_R(R \otimes G) - \operatorname{rank}_R \mathbf{d}(R \otimes G) \leq \operatorname{rank}_{\hat{W}}(\hat{W} \otimes G) - \operatorname{rank}_{\hat{W}} \mathbf{d}(\hat{W} \otimes G) = \operatorname{rank}_{\hat{W}} \hat{W} \star G$ (since $\operatorname{rank}_R R \otimes G = \operatorname{rank}_{\hat{W}} \hat{W} \otimes G = \operatorname{rank} G$). But by Theorem 6.6 $\operatorname{rank}_{\hat{W}} \hat{W} \star G =$ p-rank G. Thus $\mathbf{d}(\hat{W} \otimes G)$ has a basis in $K \otimes G$ if and only if $\operatorname{rank}_R(R \star G) \leq$ p-rank $G = (Rp)$-$\operatorname{rank}_R R \star G$, where the last equality is by Lemma 6.2. By Proposition 1.26 this inequality holds if and only if $R \star G$ is a projective R-module, i.e. if and only if G is R-split. ☑

COROLLARY 6.20. *Every W-module over a discrete valuation ring W has a splitting field which is a finitely generated extension of Q.*

PROOF: Choose a basis for $\mathbf{d}(\hat{W} \otimes G)$, and write these basis elements as linear combinations of a fixed basis for QG. If K is the field generated by the corresponding coefficients, then K is a finitely generated field extension of Q and the given basis for $\mathbf{d}(\hat{W} \otimes G)$ lies in $K \otimes G$. Hence by Proposition 6.19 K is a splitting field for G. ☑

CORNER'S THEOREM. Looking at module over a discrete valuation ring W as an intersection of a Q-vector space with a completely decomposable $\hat{W}$-module enables us to get a proof in the local case of the celebrated theorem of A.L.S. Corner asserting that every reduced finite rank torsion free ring over a discrete valuation ring which is "sufficiently incomplete" (as it were) is the endomorphism ring of some finite rank torsion free module. Once the theorem is proved in the local case, it is then easy to get the general case.

We start with a preliminary lemma.

LEMMA 6.21. *Let S be a finite rank torsion free ring over a discrete valuation ring W, let $\hat{S}$ be its completion, and let K be a splitting field for S (as a W-module). Let $R = K \cap \hat{W}$. Let $t_1, \ldots, t_n$ be elements of $\hat{W}$ algebraically independent over K. Then $t_1, \ldots, t_n$, considered as elements of the $\hat{W}$-algebra $\hat{S}$, are algebraically independant over the ring S in the sense that $1, t_1, \ldots, t_n, t_1^2, t_1 t_2, \ldots$ are linearly independent over S.*

PROOF: $\hat{S}$ is a $\hat{W}$-algebra and thus also an R-algebra. Furthermore S generates $\hat{S}$ as a $\hat{W}$-algebra. Since $1, t_1, \ldots, t_1^2, \ldots$ are elements of $\hat{W}$ linearly independent over R, then $1 \otimes 1$, $t_1 \otimes 1$, …, $t_1^2 \otimes 1$, … are elements of $\hat{W} \otimes_R RS$ linearly independent over RS, where RS denotes the R-subalgebra of $\hat{S}$ generated by S. By Lemma 6.2 $RS \approx R \star S$, and since K is a splitting field for S, $R \star S$ is a free R-module. Thus RS is a free R-module and so $\hat{W} \otimes_R RS$ is a free $\hat{W}$-module, and in particular is reduced, so that $\hat{W} \otimes_R RS \approx \hat{W} \star_R RS$. Hence by Lemma 6.2 the map $\mu \colon \hat{W} \otimes_R RS \to \hat{W}RS = \hat{W}S = \hat{S}$ such that $\mu(\hat{w} \otimes x) = \hat{w}x$ is an isomorphism. Since $\mu(1 \otimes 1) = 1$, $\mu(t_1 \otimes 1) = t_1, \ldots, \mu(t_1^2 \otimes 1) = t_1^2, \ldots$, it follows that $1, t_1, \ldots, t_1^2, t_1 t_2, \ldots$ are elements of $\hat{S}$ linearly independent over RS and hence also over S. $\boxed{\checkmark}$

PROPOSITION 6.22. *Let S be a reduced torsion free ring with rank n over a discrete valuation ring W such that $Q\hat{W}$ has infinite transcendance degree over Q. Then there exists a finite rank torsion free W-module G with* $\operatorname{rank} G = 2n$ *and* $\text{p-rank}\, G = \text{p-rank}\, S$ *such that* $\operatorname{End} G \approx S$.

PROOF: The idea is to construct G as a pure p-adically dense S-submodule of $\hat{S}$. Thus $\hat{G} = \hat{S}$ and so any endomorphism of G extends uniquely to an endomorphism of $\hat{G} = \hat{S}$. This will give an embedding $\operatorname{End} G \to \operatorname{End} \hat{S}$. And since G is an S-module, left multiplication by any $s \in S$ is an endomorphism of G so that there is a ring morphism $\rho \colon S \to \operatorname{End} G \subseteq \operatorname{End} \hat{S}$. We will construct G so that $1 \in G$, which will ensure that ρ is monic. Thus

$$S \subseteq \operatorname{End} G = \{\varphi \in \operatorname{End} \hat{S} \mid \varphi(G) \subseteq G\}.$$

By making G extremely "generic" in $\hat{S}$ we will ensure that $\operatorname{End} G = S$.

Let K be a splitting field for the W-module S and let $R = K \cap \hat{W}$. By Corollary 6.20 we can choose K to be a finitely generated field extension of Q. Then $Q\hat{W}$ has infinite transcendence degree over K. Choose elements $t_1, \ldots, t_n \in \hat{W}$ which are algebraically independent over K. Let $s_1 = 1, \ldots, s_n$ be a maximal linearly independent set in S and let

$$f = s_1 t_1 + \cdots + s_n t_n \in \hat{S}.$$

($\hat{S}$ is a $\hat{W}$-algebra, so we can identify $\hat{W}$ as a subring of the center of $\hat{S}$.) Let G be the pure closure of the S-submodule of $\hat{S}$ generated by 1 and f:

$$G = (S + Sf)_*.$$

Then $S \quad \lhd \quad G \quad \lhd \quad \hat{S}$ so that G is p-adically dense in $\hat{S}$ and also $\text{p-rank}\, S \le \text{p-rank}\, G \le \text{p-rank}\, \hat{S} = \text{p-rank}\, S$, so $\text{p-rank}\, G = \text{p-rank}\, S$. By Lemma 6.21, 1, t_1, …, t_1^2, $t_1 t_2$, … are linearly independent over S, and hence also over QS. Now if $s \in S$ and $sf = \sum s s_i t_i = 0$ then since the t_i are linearly independent over S, all $s s_i = 0$ and in particular $s = s1 = s s_1 = 0$. Thus the mapping $s \mapsto sf$ gives an isomorphism $Sf \approx S$. Furthermore it follows from the independence of $1, t_1, \ldots, t_n$ over S, that $S \cap Sf \subseteq S \cap (\bigoplus_i S t_i) = 0$. Thus $\operatorname{rank} G = \operatorname{rank}(S + Sf) = \operatorname{rank} S + \operatorname{rank} Sf = 2 \operatorname{rank} S = 2n$.

Our objective is to show that if $\varphi \in \operatorname{End} \hat{S} = \hat{S}$ and $\varphi(G) \subseteq G$ then φ is given by left multiplication by some $s \in S$. Now for each $i = 1, \ldots, n$, since $\varphi(s_i) \in G = (S + Sf)_*$ there exist s_i', $s_i'' \in QS$ such that

$$\varphi(s_i) = s_i' + s_i'' f.$$

By Proposition 3.43 φ is $\hat{W}$-linear so

$$\varphi(f) = \varphi(s_1 t_1 + \cdots + s_n t_n) = \sum \varphi(s_i) t_i = \sum (s_i' + s_i'' f) t_i = \sum s_i' t_i + \sum \sum s_i'' s_j t_i t_j.$$

Since this must belong to $QS + QSf \subseteq Q(S + St_1 + \ldots St_n)$, and since $1, t_1, \ldots, t_1^2, \ldots$ are linearly independent over QS, one concludes that the coefficients of t_i^2 and $t_i t_j$ all vanish, i.e. $(\forall i)\, s_i'' s_i = 0$ and $(\forall i \neq j)\, s_i'' s_j + s_j'' s_i = 0$. Since $s_1 = 1$ this yields $s_1'' = 0$ and so for all j, $0 = s_1'' s_j + s_j'' s_1 = s_j'' 1 = s_j''$. Thus for all i, $\varphi(s_i) = s_i' + s_i'' f = s_i'$ and $\varphi(f) = \varphi(s_1 t_1 + \cdots + s_n t_n) = s_1' t_1 + \cdots + s_n' t_n$.

Now note that $QG = QS \oplus QSf \subseteq QS \oplus \bigoplus_i QSt_i$ and $QSf \subseteq \bigoplus_i QSt_i$, so that $QG \cap (\bigoplus QSt_i) = QSf$. Thus since $\varphi(G) \subseteq G$, $\varphi(f) \in G \cap (\bigoplus QSt_i) \subseteq QSf$ so that $\varphi(f) = s_0 f$ for some $s_0 \in QS$. Thus $\varphi(f) = s_1' t_1 + \cdots + s_n' t_n = s_0 f = s_0(s_1 t_1 + \cdots + s_n t_n)$. Once again using the linear independence of the t_i over QS we get $\varphi(s_i) = s_i' = s_0 s_i$ for all i. Since $s_1, \ldots, s_n$ are a basis for QS over Q they also generate $Q\hat{S}$ over $Q\hat{W}$. It thus follows that $\varphi(g) = s_0 g$ for all $g \in \hat{S}$, and in particular for all $g \in G$. Furthermore since $S \vartriangleleft G$, $s_0 = \varphi(1) \in G \cap QS = S$. Thus every endomorphism φ of G is given by left multiplication by some $s_0 \in S$, so that $\operatorname{End} G \approx S$. $\boxed{\checkmark}$

THEOREM 6.23. *Let W be a dedekind domain such that for each prime p, $Q\hat{W}_p$ has infinite transcendence degree over Q. Let S be a reduced torsion free ring over W with rank n. Then there exists a torsion free W-module G with* $\operatorname{rank} G = 2n$ *and* $\text{p-rank}\, G = \text{p-rank}\, S$ *for all p and such that* $\operatorname{End} G \approx S$.

PROOF: Think of $\prod_p \hat{S}_p$ as an S-algebra in the obvious way and let $\delta\colon S \to \prod_p \hat{S}_p$ be the corresponding embedding, i.e. δ is the diagonal map, the product of the canonical maps $\zeta_p\colon S \to \hat{S}_p$. Then $\operatorname{Ker} \delta = \bigcap \operatorname{Ker} \zeta_p = \bigcap_p p^\infty S$, which is trivial since S is reduced, so δ is monic. Note that $\operatorname{Hom}(\hat{S}_p, \hat{S}_{p'}) = 0$ if $p' \neq p$, so that each factor $\hat{S}_p$ is fully invariant in the product $\prod_p \hat{S}_p$. Thus if $\varphi \in \operatorname{End}\left(\prod \hat{S}_p\right)$ then φ induces an endomorphism $\hat{\varphi}_p$ of $\hat{S}_p$ for each p. As in Proposition 6.22, let $s_1 = 1, \ldots, s_n$ be a maximal linearly independent set in S and for each prime p construct a W_p-submodule $(S_p + S_p f_p)_*$ of $\hat{S}_p$ such that if $\varphi \in \operatorname{End}(S_p + S_p f_p)_*$ then φ is given by left multiplication by s_{0p} for some $s_{0p} \in S_p$. (The element s_{0p} is not necessarily uniquely defined, but its image in $S_p/\mathbf{d}(S_p)$ is unique.) Let $f = (f_p)_p \in \prod_p \hat{S}_p$ and let $G = (\delta(S) + Sf)_* = (S\delta(1) + f)_* \subseteq \prod \hat{S}_p$. In other words for $g \in G$ there exist s, $s' \in QS$ such that for all p, the p^{th} coordinate of g is $s + s' f_p$. Then if $\varphi \in \operatorname{End} \prod_p \hat{S}_p$ and $\varphi(G) \subseteq G$, then $\varphi(s_i) = s_i' \oplus s_i'' f$, for $i = 1, \ldots, n$ where s_i', $s_i'' \in QS$. (More precisely, we should write $\varphi(\delta(s_i))$ rather than $\varphi(s_i)$.) Then as shown in the proof of Proposition 6.22, we see that $(s_i'')_p = 0$ for all i and it follows that $s_i'' = 0$. The proof of Proposition 6.22 then further shows that $\varphi(f) = s_0 f$ for some $s_0 \in S$ and then $\varphi(s_i) = s_0 s_i$ for all i. Since $s_1, \ldots, s_n$ are a basis for QS, they also generate $\prod_p \hat{S}_p$ over $\prod_p \hat{W}_p$. Since φ is linear over $\prod_p \hat{W}_p$, thus $\varphi(g) = s_0 g$ for all $g \in \prod_p \hat{S}_p$, and in particular for all $g \in G$. Thus there is a ring monomorphism $\operatorname{End} G \to S$ given by $\varphi \mapsto s_0$. Since G is an S-module there is an inverse morphism sending s_0 to the endomorphism of G given by left multiplication by s_0. Therefore $\operatorname{End} G \approx S$. $\boxed{\checkmark}$

SPLITTING RINGS. We can now obtain the analogs of Theorem 6.9 and Theorem 6.10 for the category of K-split modules. However once one drops the restriction $R = \hat{W}$ there is no longer much to be gained by restricting the context to modules over discrete valuation rings.

DEFINITION 6.24. A splitting ring R over a dedekind domain W is a ring (possibly with infinite rank) such that $\text{p-rank}\, R = 1$ for all primes p and QR is a finite product of fields.

If R is a splitting ring, we say that a W-module G is **R-split** if $R \star G$ is a finitely generated R-module. Note that if W is a discrete valuation ring then this agrees with our previous usage of the term R-split.

The requirement that QR be a product of fields can be omitted from the definition of a splitting ring if R has finite rank.

PROPOSITION 6.25. *If R is a reduced* finite rank *torsion free ring and* $\text{p-rank}\, R = 1$ *for all p then R is a splitting ring.*

PROOF: W is a pure subring of R by Lemma 6.2. Furthermore since R is a Murley ring, by Proposition 3.43 QR is a finite dimensional commutative Q-algebra with trivial nil-radical, hence a finite product of fields. $\boxed{\checkmark}$

By Proposition 3.43 if R is a splitting ring then R is a finite product of dedekind domains R_i, and every maximal ideal in R_i has the form pR_i where p is a prime ideal in W. In particular, if W is local then R is a discrete valuation ring with maximal ideal pR. (If W has only one prime then there could be only one factor in the product $\prod R_i$ since the R_i are reduced and $\text{p-rank} \prod R_i = 1$.)

PROPOSITION 6.26. *Let R be a splitting ring and G a finite rank torsion free W-module . The following conditions are equivalent:*

(1) *G is R-split.*
(2) *$R \star G$ is a projective R-module.*
(3) *$G/\mathbf{d}(G)$ is isomorphic to a pure submodule of a finitely generated R-module.*

PROOF: (1) $\Rightarrow$ (2): If $R = \prod_1^n R_i$ where the R_i are dedekind domains, let $e_1, \ldots, e_n$ be the primitive idempotents in this product. If a W-module G is R-split then $R \star G$ is a finitely generated torsion free R-module and it follows that for each i, $R_i \star G = e_i R \star G$ if a finitely generated R_i-module since if $x_1, \ldots, x_k$ is a finite set of generators for $R \star G$ over R then $e_i \star x_1, \ldots, e_i \star x_k$ is a finite set of generators for $R_i \star G$ as an R_i-module. Since each R_i is a dedekind domain, $R_i \star G$ is hence a projective R_i-module. It follows easily that $R \star G$ is a projective R-module.

(2) $\Rightarrow$ (3): By Proposition 1.42 $G/\mathbf{d}(G)$ is a pure submodule of $R\star G$ and if $R\star G$ is projective over R then it is finitely generated over R by Proposition 1.* and the fact that R is a product of dedekind domains.

(3) $\Rightarrow$ (1): If $G/\mathbf{d}(G)$ is a pure submodule of a finitely generated R-module M then by Proposition 1.44 and Proposition 3.43 $R\star G \approx R\star G/\mathbf{d}(G) \ \lhd\ R\star M \approx M$. Since R is noetherian, $R\star G$ is thus a finitely generated R-module. ☑

COROLLARY 6.27. *If G is R-split then* $\text{p-rank}\, G = \operatorname{rank}_R R\star G$.

PROOF: Since $R\star G$ is a projective R-module, by Proposition 1.*,
✠ $\operatorname{rank}_R R\star G = \text{p-rank}_R\, R\star G = \text{p-rank}\, R\star G = (\text{p-rank}\, R)(\text{p-rank}\, G) = \text{p-rank}\, G$. ☑

PROPOSITION 6.28. (1) *Finitely generated modules are R-split.*

(2) *A pure submodule of an R-split module is R-split.*

(3) *A homomorphic image of an R-split module is R-split.*

(4) *A module quasi-isomorphic to an R-split module is R-split.*

(5) *If G and H are R-split then so is $G\otimes H$.*

(6) *If H is R-split then so is* $\operatorname{Hom}(G,H)$.

PROOF: (1) If F is finitely generated then $R\star F$ is finitely generated over R since any set of generators for F as a W-module generates $R\star F$ as an R-module. Therefore F is R-split.

(2) If $H \lhd G$ then by Proposition 1.44 $R\star H \lhd R\star G$. If G is R-split then $R\star G$ is a finitely generated R-module, hence since R is noetherian $R\star H$ is a finitely generated R-module and so H is R-split.

(3) If K is a homomorphic image of G, then by Proposition 1.42 $R\star K$ is a homomorphic image of $R\star G$. Thus if $R\star G$ is a finitely generated R-module so is $R\star K$.

(4) If G' is quasi-isomorphic to G then G' is a homomorphic image of $G\oplus F$ for some finitely generated W-module F by Proposition 3.1. Hence if G is R-split then G' is R-split by (1) and (3).

(5) Since $R\otimes G\otimes H \approx (R\otimes G)\otimes_R (R\otimes H)$, it follows that $R\star(G\otimes H) \approx (R\star G)\star_R (R\star H)$. Thus if G and H are R-split so that $R\star G$ and $R\star H$ are finitely generated R-modules, then $R\star(G\otimes H)$ is also a finitely generated R-module, so that $G\otimes H$ is R-split.

(6) Since $H/\mathbf{d}(H) \lhd R\star H$, by Proposition 1.40 $\operatorname{Hom}(G, H/\mathbf{d}(H)) \lhd \operatorname{Hom}(G, R\star H)$. Furthermore by Proposition 1.44 $\operatorname{Hom}(G, R\star H) \approx \operatorname{Hom}_R(R\star G,\, R\star H)$, and this is a finitely generated R-module if $R\star H$ is. Thus if H is R-split then $\operatorname{Hom}(G, H/\mathbf{d}(H))$ is isomorphic to a pure submodule of a finitely generated R-module and hence by Proposition 6.26 is R-split. Since $\operatorname{Hom}(G,H) \approx \operatorname{Hom}(G, H/\mathbf{d}(H)) \oplus \operatorname{Hom}(G, \mathbf{d}(H))$ and by Proposition 1.38 $\operatorname{Hom}(G, \mathbf{d}(H))$ is divisible, it follows that $\operatorname{Hom}(G,H)$ is R-split. ☑

PROPOSITION 6.29. *Let $\mathcal{C}$ be the class of W-modules G for which there exists a splitting ring R such that G is R-split. Then $\mathcal{C}$ is closed under pure submodules, homomorphic images, and finite direct sums.*

PROOF: The assertions about pure submodules and homomorphic images have just been proved in Proposition 6.28. Now suppose that G_1 is R_1-split and G_2 is R_2-split for splitting rings R_1 and R_2. Then $\text{p-rank}\, R_1 \star R_2 = (\text{p-rank}\, R_1)(\text{p-rank}\, R_2) = 1$ for all p ✠ and $Q(R_1 \star R_2) = ****$, a finite product of fields, so $R_1 \star R_2$ is a splitting ring. Since G_1 is R_1-split, $R_1 \star G_1$ is a finitely generated R_1-module. Thus $R_1 \star R_2 \star G_1$ is a finitely generated $R_1 \star R_2$-module, so that G_1 is $R_1 \star R_2$-split. Likewise G_2 is $R_1 \star R_2$-split. Thus $G_1 \oplus G_2$ is $R_1 \star R_2$-split. $\boxed{\checkmark}$

PROPOSITION 6.30. *If there exists a Murley ring R such that QR is a finite product of fields and $R \star G$ is a finitely generated R-module and G is p-divisible for all primes p such that* $\text{p-rank}\, R = 0$, *then there exists a splitting ring R' such that G is R-split.*

PROOF: R may not be a splitting ring, since there may be primes p such that $\text{p-rank}\, R = 0$. Now let A be the subring of Q such that $\text{p-rank}\, A = 0 \iff \text{p-rank}\, R = 1$ (Proposition 2.3), and let $R' = R \oplus A$. Then $\text{p-rank}\, R' = 1$ for all p and $QR' = QR \times Q$, a finite product of fields, so R' is a splitting ring. Now for all prime ideals p, $\text{p-rank}\, A \neq 0 \Rightarrow \text{p-rank}\, R = 0 \Rightarrow \text{p-rank}\, G = 0$ Thus $\text{p-rank}\, A \otimes G = 0$ for all p so that $A \otimes G$ is divisible and $A \star G = 0$. Thus $R' \star G = (R \star G) \oplus (A \star G) = R \star G$, and since by assumption this is finitely generated over R it is also a finitely generated module over R', so G is R'-split. $\boxed{\checkmark}$

COROLLARY 6.31. *If G is p-divisible for all but finitely many primes p then G is R-split for some splitting ring R.*

PROOF: Let $R = \prod \hat{W}_p$, where the product is taken over the finite set of primes p for which G is not p-divisible. Then R is a Murley ring (probably with infinite rank) and for each p, $\hat{W}_p \star G \approx \hat{G}_p$ is a finitely generated $\hat{W}_p$-module. Thus $R \star G = \prod \hat{W}_p \star G$ is a finitely generated R-module, and so by Proposition 6.30 G is R'-split for some splitting ring R'. $\boxed{\checkmark}$

We can now justify the claim at the beginning of the Chapter that the concept of R-split modules is the key to understanding the Pontryagin construction even when W is not local.

EXAMPLE 1.47 [CONTINUED]. Let W be a dedekind domain, let p be a prime and let G be constructed by adjoining to $W \oplus W \oplus W$ the elements of $p^{-n}(v_n, 1, 0)$ and $p^{-n}(w_n, 0, 1)$ for all n, where the sequences $\{v_n\}$ and $\{w_n\}$ converge to elements in v and w in $\hat{W}_p$ such that 1, v, w are linearly independent.

Let $R = \hat{W}_p \oplus p^{-\infty}$. Then p'-rank $R = 1$ for all primes p' and $QR = Q\hat{W}_p \oplus Q$, so R is a splitting ring. Let V be the subspace of $R = \hat{W}_p \oplus p^{-\infty}$ generated by $(1,1)$, $(v,1)$, and $(w,1)$. Then $G = V \cap R$ and so G is R-split.

✠ PROOF: **** $\boxed{\checkmark}$

We can now give the promised generalizations of Theorem 6.9 and Theorem 6.10.

THEOREM 6.32. *Let R be a splitting ring. Consider the category where the objects consist of pairs (V, D) with V a finite dimensional Q-space and D a QR-subspace of $R \otimes V$, and where a morphism from (V, D) to (V', D') is a Q-linear map $\varphi\colon V \to V'$ such that $(R \otimes \varphi)(D) \subseteq D'$. Then this category is isomorphic to the category of finite rank R-split modules under quasi-homomorphisms.*

PROOF: The proof is identical to that of Theorem 6.9. There is a functor in one direction sending G to the pair $(QG, \mathbf{d}(R \otimes G))$ and a functor in the other direction sending a pair (V, D) to $V \cap (D \oplus F)$ where F is a projective R-submodule of $R \otimes V$ such that $D \oplus F$ is an essential submodule of $R \otimes V$. By Lemma 6.8, if $G = V \cap (D \oplus F)$, then $QG = V$ and $\mathbf{d}(R \otimes G) = D$, so we get back to where we started. This also shows that by Theorem 6.6, G is independent of the choice of F, up to quasi-equality.

One the other hand, if we start with G then we may as well choose F so that $\mathbf{d}(\hat{W} \otimes G) \oplus F = \hat{W} \otimes G$. (Recall that $(R \otimes G)/d(R \otimes G) = R \star G$ is projective since G is R-split.) Thus by Proposition 6.3, $V \cap (D \oplus F) = V \cap (R \otimes G) = G$. Thus the two functors are inverse to each other. $\boxed{\checkmark}$

THEOREM 6.33. *Let R be a splitting ring. Consider the category where the objects consist of pairs (L, V) with L a finitely generated R-module and V a Q-subspace of QL such that V generates QL as an R-module, and where a morphism from (L, V) to (L', V') consists of a R-linear map $\psi\colon L \to L'$ such that $\psi(V) \subseteq V'$. Then this category is isomorphic to the category of reduced finite rank R-split modules under homomorphisms.*

PROOF: If G is a reduced W-module then QG generates $Q(R \star G)$ as a R-module. Thus we can associate with G the pair $(R \star G, QG)$ in $\mathcal{C}$. Conversely, given a pair (L, V) in $\mathcal{C}$ we get a W-module $G = V \cap L$.

First, note that these correspondences are functors. In fact, if $\varphi\colon G \to H$ then φ induces maps $QG \to QH$ and $R \star G \to R \star H$. And if $\psi\colon (L, V) \to (L', V')$ then by definition ψ is a map $L \to L'$ such that $\psi(V) \subseteq V'$, so clearly ψ induces a map $V \cap L \to V' \cap L'$.

Now if we start with G and go up and then come back down, we get back to $QG \cap (R \star G) = G$ because by Proposition 1.42, G is pure in $R \star G$. Thus the composition of the two functors that way is the identity.

Now suppose we start with a pair (L, V) such that $RV = QL$ and set $G = V \cap L$. ✠ Then $QG = V$ and RG is an essential W-submodule of L (WHY?), so by Lemma 6.7, $RG = L$. But by Lemma 6.2, $R \star G \approx RG$. Thus there is an isomorphism $\beta\colon R \star G \approx L$ which is the identity on $QG = V$. Hence β yields an isomorphism from the pair $(R \star G, QG)$ to (L, V), showing that the composition of the two functors in this case is also naturally equivalent to the identity.

Thus the two functors are inverse to each other and hence give an equivalence between the two categories. $\boxed{\checkmark}$

REMARK: For categorical constructions, the restriction that G is reduced can be a nuisance, since a homomorphic image of a reduced module need not be reduced. The most categorical remedy for this is to let the category for Theorem 6.33 consist of triples

(L, V, λ), where L is a finitely generated R-module, V a Q-vector space, and $\lambda\colon V \to QL$ is a map such that $\lambda(V)$ generates QL as an R-module. A morphism in this category from (L_1, V_1, λ_1) to (L_2, V_2, λ_2) then consists of a pair of maps $\varphi\colon V_1 \to V_2$ and $\psi\colon L_1 \to L_2$ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\lambda_1"'] & V_2 \arrow[d, "\lambda_2"'] \\ QL_1 \arrow[r, "\psi"] & QL_2. \end{tikzcd}$$

The functor in one direction then takes an R-split W-module G to the triple $(R \star G,\, QG,\, \lambda)$, where $\lambda\colon G \to R \star G$ is the canonical map. And the other functor takes a triple (L, V, λ) to $\lambda^{-1}(L)$. Note that if $G = \lambda^{-1}(L)$ then since L is reduced, $\mathbf{d}(G) = \operatorname{Ker} \lambda$. The construction can also be described by the following pull-back diagram:

$$\begin{tikzcd} G \arrow[r, "\subseteq"] \arrow[d] & V \arrow[d, "\lambda"'] \\ L \arrow[r, "\subseteq"] & QL. \end{tikzcd}$$

If G is reduced then λ is monic and it is easy to see that this reduces to the construction in Theorem 6.33.

BUTLER MODULES. At the beginning of this chapter the class of modules obtained by constructions analogous to the Pontryagin module was contrasted with the class of Butler modules. Now, though, we will see that a large class of Butler modules – namely those whose typesets contain only idempotent types – is contained within the class of modules which are R-split for a finite rank splitting ring R.

PROPOSITION 6.34. *A finite rank torsion free module G is a Butler module such that $\mathbf{T}(G)$ contains only idempotent types if and only if G is R-split for some splitting ring R which is a finite product of subrings of Q.*

PROOF: ($\Rightarrow$): If G is a Butler module then by Proposition 5.* G is isomorphic to a pure submodule of a direct sum of rank-one modules whose types belong to $\mathbf{CT}(G)$. Thus by Proposition 6.29 it suffices to prove that there exists a splitting ring R which is a finite product of subrings of Q such that every rank-one module A with $\mathbf{t}(A) \in \mathbf{CT}(G)$ is R-split. By Proposition 2.3 the set of idempotent types forms a complemented lattice and by Proposition 5.* if G is a Butler module then $\mathbf{CT}(G)$ is finite and contained in the lattice generated by $\mathbf{T}(G)$. It follows that if $\mathbf{T}(G)$ containes only idempotent types then $\mathbf{CT}(G)$ is contained in a finite complemented lattice of idempotent types. Let $R_1, \ldots, R_n$ be a set of subrings of Q representing the maximal proper (non-divisible) types in this complemented lattice. Thus $R_i + R_j = Q$ for $i \neq j$ and $\bigcap_1^n R_i = W$ and every $\mathbf{t} \in \mathbf{CT}(G)$ is the greatest lower bound of some subset of $\{\mathbf{t}(R_1), \ldots, \mathbf{t}(R_n)\}$. Hence for every prime p there is exactly one i such that p-rank $R_i \neq 0$ and it follows that if $R = \prod R_i$ then $\text{p-rank}\, R = 1$, so that R is a splitting ring. Furthermore if $\mathbf{t} \in \mathbf{CT}(G)$ then for any i either $\mathbf{t} \leq \mathbf{t}(R_i)$ and so $\mathbf{t}(R_i) \vee \mathbf{t} = \mathbf{t}(R_i)$ or else $\mathbf{t}(R_i) \vee \mathbf{t} = \mathbf{t}(Q)$. Since by Proposition 2.18 $\mathbf{t}(R_i) \vee \mathbf{t} = \mathbf{t}(R_i)\mathbf{t}$, this says that if A is a subring of Q with $\mathbf{t}(A) \in \mathbf{CT}(G)$ then either $R_i \star A = R_i A = R_i$ or $R_i \star A = 0$, so that $R \star A = \prod' R_i$, where the right-hand product is taken over the subset of the R_i with $R_i \star A \neq 0$. Thus $R \star A$ is a finitely generated R-module, so that A is R-split. Since this is true for every rank-one module A with $\mathbf{t}(A) \in \mathbf{CT}(G)$ and since G is isomorphic to a pure submodule of a direct sum of such rank-one modules, it follows from Proposition 6.28 that G is R-split.

($\Leftarrow$): If R_i is a subring of Q and $\mathbf{t}_i = \mathbf{t}(R_i)$ then $\mathbf{t}_i$ is idempotent and by Proposition 2.2(?) R_i is a dedekind domain so that a finitely generated R_i-module is a projective R_i-module and hence by Proposition 4.11 is $\mathbf{t}_i$-projective. Thus if $R = \prod R_i$ then a finitely generated R-module is a direct sum of $\mathbf{t}_i$-projective modules and in particular is completely decomposable and by Corollary 2.24 the types in its typeset are all idempotent. Now if G is R-split then G is isomorphic to a pure submodule of a finitely generated R-module, so G is a pure submodule of a completely decomposable module, so G is a Butler module and $\mathbf{T}(G)$ contains only idempotent types. ☑

EXTENSION OF SCALARS. We now see how the splitting ring for a W-module G is related to the splitting ring for the W'-module $W' \otimes G$ where W' is a finite integral extension of W.

LEMMA 6.35. *Let W' be a ring which is a finite rank projective W-module. Then if G and H are finite rank torsion free W-modules,*

(1) $\operatorname{Hom}_{W'}(W' \otimes G,\, W' \otimes H) \approx W' \otimes \operatorname{Hom}(G, H)$.

(2) $G \sim H \iff W' \otimes G \sim W' \otimes H$.

PROOF: (1) The W'-linear map $W' \otimes \operatorname{Hom}(G,H) \to \operatorname{Hom}_{W'}(W' \otimes G,' W' \otimes H)$ given by $1 \otimes \varphi \mapsto 1_{W'} \otimes \varphi$ can be factored as

$$W' \otimes \operatorname{Hom}(G,H) \xrightarrow{\eta} \operatorname{Hom}(G,\, W' \otimes H) \to \operatorname{Hom}_{W'}(W' \otimes G,\, W' \otimes H)$$

where the map η is given by $(\eta(w' \otimes \varphi))(g) = w' \otimes \varphi(g)$ and the second map is the canonical adjointness isomorphism. The map η is also an isomorphism. In fact, if P is any projective W-module then the map $\eta\colon P \otimes \operatorname{Hom}(G,H) \to \operatorname{Hom}(G, P \otimes H)$ given by $(\eta(p \otimes \varphi))(g) = p \otimes \varphi(g)$ is an isomorphism. This is clear for the case $P = W$, and it then follows easily for finite rank free modules, and thus for summands of finite rank free modules, i.e. finite rank projective modules.

(2) ($\Rightarrow$): Clear.

($\Leftarrow$): Since W' is projective, $W' \otimes G$ and $W' \otimes H$ are quasi-isomorphic to direct sums of copies of G and H. Thus if $W' \otimes G \sim W' \otimes H$ then $G^k \approx H^k$ for some k and it follows from Jónsson's Theorem (Theorem 3.24) that $G \sim H$. ☑

LEMMA 6.36. *Let W' be a torsion free integral domain. Let p' be a maximal ideal in W' and $p = p' \cap W$. Let G be a finite rank torsion free W-module. Then* $p'\text{-rank}_{W'}(W' \otimes G) = \text{p-rank}\, G$.

PROOF: Since $p(W'/p') = 0$, $W'/p' \otimes G = W'/p' \otimes G/pG$. Furthermore if V is any vector space over W/p then since $W'/p' \otimes V = W'/p' \otimes_{W/p} V$ it is well known that $\dim_{W'/p'} W'/p' \otimes V = \dim_{W/p} V$. Thus we get

$$\begin{aligned} p'\text{-rank}_{W'} W' \otimes G &= \dim_{W'/p'} (W' \otimes G)/p'(W' \otimes G) \\ &= \dim_{W'/p'} (W'/p') \otimes_{W'} W' \otimes G \\ &= \dim_{W'/p'} (W'/p') \otimes_{W} G \\ &= \dim_{W'/p'} (W'/p') \otimes G/pG \\ &= \dim_{W/p} G/pG = \text{p-rank}\, G. \quad \boxed{\checkmark} \end{aligned}$$

Recall that that a ring W' is called a **finite integral extension** of W if W' is a finitely generated W-module.

PROPOSITION 6.37. *Let R be a splitting ring and let W' be a finite rank torsion free dedekind domain. Then $W' \star R$ is a splitting ring in the category of finite rank torsion free W'-modules. If, in addition, W' is a finite integral extension of W then*

1. *A finite rank W-module G is R-split if and only if the W'-module $W' \otimes G$ is $(W' \star R)$-split.*
2. *A finite rank W'-module H is $(W' \star R)$-split as a W'-module if and only if it is R-split as a W-module.*

PROOF: By Lemma 6.36 $p'\text{-rank}_{W'}\ W' \star R = \text{p-rank}\, R = 1$ for all prime ideals p' of W'.
✠ Proposition 6.25 [**This assumes finite rank.**].

(2) Since $(W' \otimes R) \otimes_{W'} H \approx R \otimes W' \otimes_{W'} H \approx R \otimes H$ it follows that $(W' \star R) \star_{W'} H \approx R \star H$, where this is an isomorphism of W'-modules. (The W'-module structure on $R \star H$ is given by $w'(r \star h) = r \star w'h$.) Now if $R \star H$ is a finitely generated R-module then surely it is finitely generated as a $W' \star R$-module. On the other hand, by hypothesis W' is a finitely generated W-module. If $w'_1, \dots, w'_k$ generate W' and $R \star H$ is generated over $W' \star R$ by the finite set $x_1, , \dots, x_\ell$, then $R \star H$ is generated over W by the finite set $w'_1 x_1, w'_2 x_1, \dots, w'_k x_\ell$. Therefore H is R-split as a W-module if and only if it is $(W' \star R)$-split as a W'-module.

(1) $W' \otimes G$ is quasi-isomorphic to a direct sum of copies of G. Thus G is an R-split W-module if and only if $W' \otimes G$ is an R-split W-module. And by (2) $W' \otimes G$ is R-split as a W-module if and only it is $W' \otimes R$-split as a W'-module.

ARNOLD DUALITY (LOCAL CASE). An important construction in linear algebra is that of the dual space. If U and V are vector spaces and $\varphi\colon U \to V$ a linear transformation, then there exist vector spaces U^* and V^* and a linear transformation $\varphi^*\colon V^* \to U^*$ such that the correspondence $V \mapsto V^*$ and $\varphi \mapsto \varphi^*$ is a contravariant functor and V^{**} is naturally equivalent to V. The functor $V \mapsto V^*$ is, of course, simply $\text{Hom}(\_\_, Q)$.

It turns out that the dual space is a construction in which carries over to the category of modules which are R-split for some splitting ring R. In fact, we are going to define a contravariant functor $\mathbf{A}$ called the **Arnold dual** for R-split modules with the property that $Q\mathbf{A}G = \text{Hom}(G, Q)$ and $\text{QHom}(\mathbf{A}H, \mathbf{A}G)$ is naturally isomorphic to $\text{QHom}(G, H)$.

(Initially we will fix R, but later we will show that $\mathbf{A}$ is independent of R.) We will not need to assume that R has finite rank, so in particular the construction below is valid for all modules over a discrete valuation ring W.

We have seen in Theorem 6.32 that a quasi-equality class G of R-split W-modules is determined by a finite dimensional vector space QG and a QR-subspace D of $R \otimes QG$, where $D = \mathbf{d}(R \otimes G)$. Starting with QG and $\mathbf{d}(R \otimes G)$ we will construct a new finite rank torsion free module $\mathbf{A}G$ by specifying the Q-space $Q\mathbf{A}G$ and a QR-subspace of $R \otimes Q\mathbf{A}G$ which will be $\mathbf{d}(R \otimes \mathbf{A}G)$.

NOTATION 6.38 If G is any finite rank torsion free , we will write

$$(R \otimes G)^\star = \mathrm{Hom}_R(R \otimes G,\, QR).$$

Furthermore we will write

$$\mathbf{d}(R \otimes G)^\perp = \{\psi \in (R \otimes G)^\star \mid \psi(\mathbf{d}(R \otimes G)) = 0\}.$$

(Note that these are QR-modules.) We define natural maps

$$\begin{aligned} &\delta\colon\ \mathrm{Hom}(G, QR) \to (R \otimes G)^\star = \mathrm{Hom}_R(R \otimes G,\, QR) \\ &\eta\colon R \otimes \mathrm{Hom}(G, Q) \to (R \otimes G)^\star \end{aligned}$$

by setting $\delta(\psi)(r \otimes g) = r\psi(g) \in QR$ and $\eta(r' \otimes \varphi)(r \otimes g) = r'r\varphi(g) \in QR$.

PRELIMINARY DESCRIPTION OF THE CONSTRUCTION. By Theorem 6.32, $\mathbf{A}G$ will be determined up to quasi-equality by its divisible hull $Q\mathbf{A}G$ and by $\mathbf{d}(R \otimes \mathbf{A}G)$. We will set $Q\mathbf{A}G = \mathrm{Hom}(G, Q)$. The natural isomorphism η enables us to identify $R \otimes \mathrm{Hom}(G, Q)$ with $\mathrm{Hom}_R(R \otimes G,\, QR)$. In other words, one can identify $R \otimes Q\mathbf{A}G$ with $(R \otimes G)^\star$. Using this identification, we set $\mathbf{d}(R \otimes \mathbf{A}G) = \mathbf{d}(R \otimes G)^\perp$. It is then easy to see that if $\varphi \in \mathrm{QHom}(G, H)$ and one defines $\mathbf{A}\varphi = \varphi^*\colon \mathrm{Hom}(H, Q) \to \mathrm{Hom}(G, Q)$, then the canonical extension of $\mathbf{A}\varphi$ to a map $(R \otimes H)^\star \to (R \otimes G)^\star$ maps $\mathbf{d}(R \otimes H)^\perp$ into $\mathbf{d}(R \otimes G)^\perp$ and thus by Theorem 6.32, $\mathbf{A}\varphi \in \mathrm{QHom}(\mathbf{A}H, \mathbf{A}G)$.

Finally, one sees that if $\rho\colon QG \to \mathrm{Hom}(\mathrm{Hom}(QG, Q),\, Q)$ is the usual duality isomorphism, then ρ yields a quasi-isomorphism from G to $\mathbf{A}^2G$.

Many will find the above description quite adequate. The remainder of this section will be devoted to spelling out the details for those who have qualms as well as proving that $\mathrm{QHom}(\mathbf{A}G, H) \approx \mathbf{d}(G \otimes H)$.

The following lemma will be used several times in this section.

LEMMA 6.39. (1) *If P is a finitely generated projective module over any ring S then for any S-module M there is an isomorphism*

$$\zeta\colon P \otimes_S M \to \mathrm{Hom}_S(\mathrm{Hom}_S(P, S),\, M)$$

given by $\zeta(p \otimes m)(\varphi) = \varphi(p)m$.

(2) *If either* P *or* M *is a finitely generated projective module over* S *then there is an isomorphism*

$$\zeta' \colon P \otimes_S \operatorname{Hom}_S(M, N) \to \operatorname{Hom}_S(M, P \otimes_S N)$$

given by $\zeta'(p \otimes \varphi)(m) = p \otimes \varphi(m)$.

PROOF: $\boxed{\checkmark}$

The reader is reminded that the conventions of this book identify $\operatorname{QHom}(G, R)$ as a submodule of $\operatorname{Hom}(G, QR)$. $\operatorname{QHom}(G, R)$ is almost always considerably smaller than $\operatorname{Hom}(G, QR)$, and the distinction between the two is fundamental in what follows.

PROPOSITION 6.40. *The natural maps*

$$\begin{aligned} \delta &\colon \operatorname{Hom}(G, QR) \longrightarrow (R \otimes G)^\star \\ \eta &\colon R \otimes Q\mathbf{A}G \longrightarrow (R \otimes G)^\star \end{aligned}$$

given above in Notation 6.38 are both isomorphisms.

Furthermore, $\delta(\operatorname{QHom}(G, R)) = \mathbf{d}(R \otimes G)^\perp$.

PROOF: First, $\delta \colon \operatorname{Hom}(G, QR) \approx \operatorname{Hom}_R(R \otimes G,\, QR)$ is simply the well known change of rings isomorphism (Proposition 0.*). Now from Lemma 6.39 we get a natural isomorphism

$$\eta' \colon R \otimes \operatorname{Hom}(G, Q) \xrightarrow{\approx} \operatorname{Hom}(G, QR)$$

given by $\eta'(r \otimes \varphi)(g) = r\varphi(g)$. To see that this follows from Lemma 6.39, note that since $\operatorname{Hom}(G, Q)$ is divisible, $R \otimes \operatorname{Hom}(G, Q) = QR \otimes \operatorname{Hom}(QG, Q)$ and $\operatorname{Hom}(G, QR) = \operatorname{Hom}(QG, QR)$, so that really we are dealing with finite dimensional vector spaces over Q, i.e. finitely generated projective Q-modules. Now note that

$$\eta = \delta\eta' \colon R \otimes Q\mathbf{A}G = R \otimes \operatorname{Hom}(G, Q) \xrightarrow{\approx} \operatorname{Hom}_R(R \otimes G,\, QR) = (R \otimes G)^\star.$$

Now by Theorem 6.32, if $\psi \in \operatorname{Hom}(G, QR)$ then $\psi \in \operatorname{QHom}(G, R)$ if and only if $(1_R \otimes \psi)(\mathbf{d}(R \otimes G)) \subseteq \mathbf{d}(R \otimes R)$. This is equivalent to the assertion that
✠ $\delta(\psi) \in \mathbf{d}(R \otimes G)^\perp$. $\boxed{\checkmark}$

THEOREM 6.41. *There exists a contravariant functor from the category of R-split modules under quasi-homomorphisms into itself such that if* G *is an R-split module then* $Q\mathbf{A}G = \operatorname{Hom}(G, Q)$ *and there is a commutative diagram as follows:*

$$\begin{tikzcd}
\mathbf{d}(R \otimes \mathbf{A}G) \arrow[r, "\approx"] \arrow[d, "\subseteq"] & \mathbf{d}(R \otimes G)^\perp \arrow[d, "\subseteq"] & \operatorname{QHom}(G, R) \arrow[l, "\approx"'] \arrow[d, "\subseteq"] \\
R \otimes Q\mathbf{A}G \arrow[r, "\eta"] \arrow[d, equal] & (R \otimes G)^\star \arrow[d, equal] & \operatorname{Hom}(G, QR) \arrow[l, "\delta"'] \arrow[d, equal] \\
R \otimes \operatorname{Hom}(G, Q) \arrow[r, "\approx"] & \operatorname{Hom}_R(R \otimes G,\, QR) & \operatorname{Hom}(G, QR). \arrow[l, "\approx"']
\end{tikzcd}$$

Furthermore this diagram uniquely defines $\mathbf{A}G$ *up to quasi-equality and if* K *is another* R*-split module and* $\gamma\colon G \to K$, *then* $\mathbf{A}\gamma\colon \mathbf{A}K \to \mathbf{A}G$ *is given by the map* $\gamma^*\colon \operatorname{Hom}(K,Q) \to \operatorname{Hom}(G,Q)$, *where* $\gamma^*(\varphi) = \varphi\gamma$.

PROOF: By Theorem 6.32, in order to construct $\mathbf{A}G$ one needs to specify $Q\mathbf{A}G$ and $\mathbf{d}(R \otimes \mathbf{A}G)$. We define $Q\mathbf{A}G = \operatorname{Hom}(G,Q)$ and define $\mathbf{d}(R \otimes \mathbf{A}G) = \eta^{-1}(\mathbf{d}(R \otimes G)^{\perp})$. (The top horizontal arrows in the diagram are simply the restrictions of η and δ as indicated.) By Theorem 6.32 this determines an R-split module $\mathbf{A}G$ uniquely up to quasi-equality.

If $\gamma\colon G \to K$, define $\mathbf{A}\gamma = \gamma^* : Q\mathbf{A}K \to Q\mathbf{A}G$, where $\gamma^*(\varphi) = \varphi\gamma$. By Theorem 6.32, in order to see that $\mathbf{A}\gamma \in \operatorname{QHom}(\mathbf{A}K, \mathbf{A}G)$ one needs to see that $R \otimes \mathbf{A}\gamma$ maps $\mathbf{d}(R \otimes \mathbf{A}K)$ into $\mathbf{d}(R \otimes \mathbf{A}G)$. Referring to the above diagram and using the natural isomorphism $\delta^{-1}\eta$, one sees that it suffices to see that if $\psi \in \operatorname{QHom}(K,R)$ and $\gamma \in \operatorname{QHom}(G,K)$, the $\gamma^*(\varphi) = \varphi\gamma \in \operatorname{QHom}(G,R)$. This is true since the composition of quasi-homomorphisms is a quasi-homomorphism. $\boxed{\checkmark}$

ALTERNATE CONSTRUCTION OF $\mathbf{A}G$: Think of $\operatorname{Hom}(G,Q)$ as sitting inside $\operatorname{Hom}(G,QR)$. Then choose a free R-submodule F of $\operatorname{Hom}(G,QR)$ such that

$$\operatorname{Hom}(G,QR) = QF \oplus \operatorname{QHom}(G,R).$$

Then

$$\mathbf{A}G = \operatorname{Hom}(G,Q) \cap (F \oplus \operatorname{QHom}(G,R)).$$

PROOF: It is clear from the diagram in Theorem 6.41 that the module so constructed is $\mathbf{A}G$. $\boxed{\checkmark}$

PROPOSITION 6.42. (1) $\operatorname{rank} \mathbf{A}G = \operatorname{rank} G$.

(2) *For every prime* p, $\text{p-rank}\, \mathbf{A}G = \operatorname{rank} G - \text{p-rank}\, G$.

PROOF: (1) This is clear since $\operatorname{rank} \mathbf{A}G = \dim \operatorname{Hom}(QG,Q) = \dim QG = \operatorname{rank} G$.

(2) Since G and $\mathbf{A}G$ are R-split, Corollary 6.27 yields

$$\begin{aligned}
\text{p-rank}\, \mathbf{A}G &= \operatorname{rank}_R R \star \mathbf{A}G = \operatorname{rank}_R (R \otimes \mathbf{A}G) - \operatorname{rank}_R \mathbf{d}(R \otimes \mathbf{A}G) \\
&= \operatorname{rank} \mathbf{A}G - \operatorname{rank}_R \operatorname{QHom}(G,R) \\
&= \operatorname{rank} \mathbf{A}G - \operatorname{rank}_R \operatorname{Hom}_R(R \star G, R) \\
&= \operatorname{rank} G - \operatorname{rank}_R R \star G = \operatorname{rank} G - \text{p-rank}\, G. \quad \boxed{\checkmark}
\end{aligned}$$

PROPOSITION 6.43. *Let* G *be an* R*-split module and* H *any torsion free* W*-module. Then* (1) *Any map* $\gamma\colon Q\mathbf{A}G = \operatorname{Hom}(G,Q) \to QH$ *extends uniquely to an* R*-linear map* $\gamma_R\colon \operatorname{Hom}(G,QR) \to QR \otimes H$.

(2) $\operatorname{QHom}(\mathbf{A}G, H)$ *consists of those* $\gamma \in \operatorname{Hom}(Q\mathbf{A}G, QH)$ *such that* $\gamma_R(\operatorname{QHom}(G,R)) \subseteq \mathbf{d}(R \otimes H)$.

(3) *Let* $\mu\colon \operatorname{Hom}(\operatorname{Hom}(G,Q), QH) \to \operatorname{Hom}_R(\operatorname{Hom}(G,R), QR \otimes H)$ *be the map taking* γ *to the restriction of* γ_R *to* $\operatorname{Hom}(G,R)$. *Then* $\operatorname{QHom}(\mathbf{A}G, H) = \mu^{-1}(\mathbf{d}(\operatorname{Hom}_R(\operatorname{Hom}(G,R), R \otimes H)))$.

PROOF: This is apparent from the commutative diagram in Theorem 6.41.

(1) Any map $\gamma\colon Q\mathbf{A}G \to QH$ extends uniquely to an R-linear map $R\otimes\gamma\colon R\otimes Q\mathbf{A}G \to R\otimes QH$. Thus if δ and η are as in the commutative diagram in Theorem 6.41 then γ extends uniquely to $\gamma_R = (R\otimes\gamma)\eta\delta^{-1}$.

(2) By Theorem 6.32, $\gamma \in \mathrm{QHom}(G,H)$ if and only if $R\otimes\gamma$ maps $\mathbf{d}(R\otimes\mathbf{A}G)$ into $\mathbf{d}(R\otimes H)$. The commutative diagram in Theorem 6.41 makes it clear that this is the case if and only if γ_R maps $\mathrm{QHom}(G,R)$ into $\mathbf{d}(R\otimes H)$.

(3) By (2), $\mathrm{QHom}(\mathbf{A}G,H)$ is the inverse image

$$\{\psi\in\mathrm{Hom}_R(\mathrm{Hom}(G,QR),\, QR\otimes H) \mid \psi(\mathrm{QHom}(G,R))\subseteq\mathbf{d}(R\otimes H)\}$$

under the mapping $\gamma\mapsto\gamma_R$. Thus if we identify (as usual) $\mathrm{Hom}(\mathrm{QHom}(G,R),\,\mathbf{d}(R\otimes H))$ as a submodule of $\mathrm{Hom}(\mathrm{QHom}(G,R),\, R\otimes H)$ then by Proposition 1.*

$$\begin{aligned}\mathrm{QHom}(\mathbf{A}G,H) &= \mu^{-1}(\mathrm{Hom}_R(\mathrm{QHom}(G,R),\,\mathbf{d}(R\otimes H)))\\ &= \mu^{-1}(\mathrm{Hom}_R(\mathrm{Hom}(G,R),\,\mathbf{d}(R\otimes H)))\\ &= \mu^{-1}(\mathbf{d}(\mathrm{Hom}_R(\mathrm{Hom}(G,R),\,R\otimes H))). \quad \boxed{\checkmark}\end{aligned}$$

We now show that $\mathbf{A}^2$ is naturally equivalent to the identity functor.

PROPOSITION 6.44. *Let* $\rho\colon QG \to Q\mathbf{A}^2(G) = \mathrm{Hom}(\mathrm{Hom}(G,Q),\,Q)$ *be given by* $\rho(g)(\varphi)=\varphi(g)$. *Then* ρ *is a natural quasi-isomorphism from* G *to* $\mathbf{A}^2(G)$.

PROOF: It is well known from linear algebra that $QG \to Q\mathbf{A}^2G$ is an isomorphism of vector spaces. Therefore by Proposition 6.32 it suffices to show that $(R\otimes\rho)(\mathbf{d}(R\otimes G)) = \mathbf{d}(R\otimes\mathbf{A}^2G)$.

Applying Theorem 6.41 to G and to $\mathbf{A}G$ yields a commutative diagram

$$\begin{tikzcd}
\mathbf{d}(R\otimes\mathbf{A}^2G) \arrow[r, "\approx"] \arrow[d, "\subseteq"'] & \mathbf{d}(R\otimes\mathbf{A}G)^{\perp} \arrow[r, "\approx"] \arrow[d, "\subseteq"'] & \mathbf{d}(R\otimes G)^{\perp\perp} \arrow[d, "\subseteq"'] \\
R\otimes Q\mathbf{A}^2G \arrow[r, "\eta_{\mathbf{A}G}"] & (R\otimes Q\mathbf{A}G)^{\star} \arrow[r, "\eta_G^{\star\,-1}"] & (R\otimes G)^{\star\star}.
\end{tikzcd}$$

Now let $\rho_1\colon QR\otimes G \to (R\otimes G)^{\star\star}$ be $\eta_G^{\star\,-1}\eta_{\mathbf{A}G}(R\otimes\rho)$. Then ρ_1 is an isomorphism and it suffices to see that $\rho_1(\mathbf{d}(R\otimes G)) = \mathbf{d}(R\otimes G)^{\perp\perp}$. But since $\rho_1(x)(\psi)=\psi(x)$, this is straight linear algebra. In fact, if $x\in\mathbf{d}(R\otimes G)$ then for any $\psi\in(\mathbf{d}(R\otimes G))^{\perp}$, $\rho_1(x)(\psi)=\psi(x)=0$, so that $\rho_1(x)\in\mathbf{d}(R\otimes G)^{\perp\perp}$. On the other hand, since QR is a product of fields, it is easy to see that if $x\notin\mathbf{d}(R\otimes G)$ then there exists $\psi\in(R\otimes G)^{\star}$ such that $\psi(\mathbf{d}(R\otimes G))=0$ and $\psi(x)\neq 0$. Thus $\psi\in\mathbf{d}(R\otimes G)^{\perp}$ and $\rho_1(x)(\psi)\neq 0$, so $\rho_1(x)\notin\mathbf{d}(R\otimes G)^{\perp\perp}$. Thus ρ_1 maps $\mathbf{d}(R\otimes G)$ isomorphically onto $\mathbf{d}(R\otimes G)^{\perp\perp}$ and it follows that $R\otimes\rho$ maps $\mathbf{d}(R\otimes G)$ isomorphically onto $\mathbf{d}(R\otimes\mathbf{A}^2G)$, and by Theorem 6.32 this shows that ρ is a quasi-isomorphism. $\boxed{\checkmark}$

COROLLARY 6.45. *If G and K are R-split modules then* $\mathrm{Hom}(\mathbf{A}G, \mathbf{A}K) \approx \mathrm{Hom}(K, G)$.

PROOF: Consider the chain of maps

$$\mathrm{QHom}(G, K) \to \mathrm{QHom}(\mathbf{A}K, \mathbf{A}G) \to \mathrm{QHom}(\mathbf{A}^2G, \mathbf{A}^2K) \to \mathrm{QHom}(\mathbf{A}^3K, \mathbf{A}^3G),$$

where in each case the map is given by $\varphi \mapsto \mathbf{A}(\varphi)$. Since by Proposition 6.44 ρ is a natural isomorphism from the identity functor to $\mathbf{A}^2$, it follows that the function $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathbf{A}^2G, \mathbf{A}^2K)$ (or from $\mathrm{QHom}(\mathbf{A}K, \mathbf{A}G) \to \mathrm{QHom}(\mathbf{A}^3K, \mathbf{A}^3G)$) given by $\varphi \mapsto \mathbf{A}^2(\varphi)$ is an isomorphism. Thus the composition of each successive pair in the chain above is an isomorphism, and it follows that $\mathrm{QHom}(\mathbf{A}K, \mathbf{A}G) \to \mathrm{QHom}(\mathbf{A}(\mathbf{A}K), \mathbf{A}(\mathbf{A}G))$ is an isomorphism. Since every R-split module is isomorphic $\mathbf{A}G$ for some G, (namely, for $G = \mathbf{A}^2G$), the result follows. $\boxed{\checkmark}$

PROPOSITION 6.46. *The functor* $\mathbf{A}$ *is independent of the choice of splitting ring* R.

PROOF: Let R_1 and R_2 be splitting rings for G and let $R = R_1 \star R_2$. Then by Proposition ** R is also a splitting ring and G is R-split. It suffices to prove that if $\mathbf{A}_1G$ is the Arnold dual of G constructing using R_1 and $\mathbf{A}G$ is the Arnold dual constructing by means of R then $\mathbf{A}_1G$ and $\mathbf{A}G$ are quasi-equal. Now $Q\mathbf{A}_1G = Q\mathbf{A}G = \mathrm{Hom}(G, Q)$, so by Theorem 6.32 it suffices to prove that $\mathbf{d}(R \otimes \mathbf{A}_1G) = \mathbf{d}(R \otimes \mathbf{A}G)$. Now note that $R \otimes \mathbf{A}_1G = R \otimes_{R_1} R_1 \otimes \mathbf{A}_1G$. Furthermore, since $\mathbf{A}_1G$ is R_1-split, $R_1 \otimes \mathbf{A}_1G$ is the direct sum of $\mathbf{d}(R_1 \otimes \mathbf{A}_1G)$ and a projective R_1-module, and it follows that $\mathbf{d}(R \otimes_{R_1} (R_1 \otimes \mathbf{A}_1G)) = R \otimes_{R_1} \mathbf{d}(R_1 \otimes \mathbf{A}_1G)$. Now by Theorem 6.41 there is a commutative diagram

$$\begin{tikzcd}
R \otimes_{R_1} R_1 \otimes Q\mathbf{A}_1G \arrow[r] \arrow[d] & R \otimes Q\mathbf{A}G \arrow[d] \\
R \otimes_{R_1} \mathrm{Hom}(G, QR_1) \arrow[r, "\zeta"] & \mathrm{Hom}(G, QR)
\end{tikzcd}$$

and the vertical maps take $R \otimes_{R_1} \mathbf{d}(R_1 \otimes \mathbf{A}_1G))$ and $\mathbf{d}(R \otimes \mathbf{A}G)$ to $R \otimes_{R_1} \mathrm{QHom}(G, R_1)$ and $\mathrm{QHom}(G, R)$. Therefore it suffices to see that the isomorphism ζ maps $R \otimes_{R_1} \mathrm{Hom}(G, R_1)$ onto $\mathrm{Hom}(G, R)$. But this is true since the restriction of ζ to ✠ $R \otimes_{R_1} \mathrm{Hom}(G, R_1)$ can be obtain as the composition of the chain of isomorphisms

$$\begin{aligned}
R \otimes_{R_1} \mathrm{Hom}(G, R_1) &\approx R \otimes_{R_1} \mathrm{Hom}_{R_1}(R_1 \star G,\ R_1) \\
&\approx \mathrm{Hom}_{R_1}(R_1 \star G,\ R \otimes_{R_1} R_1) \\
&\approx \mathrm{Hom}_{R_1}(R_1 \star G,\ R) \\
&\approx \mathrm{Hom}(G, R),
\end{aligned}$$

using Proposition 6.39 and Proposition 1.*. $\boxed{\checkmark}$

THEOREM 6.47. *For any R-split module G and any torsion free W-module H there exists an isomorphism*

$$\sigma\colon QG \otimes H \to \mathrm{Hom}(\mathrm{Hom}(G, Q),\, QH) = \mathrm{Hom}(\mathbf{A}G, QH)$$

given by $\sigma(g \otimes h)(\varphi) = \varphi(g)h$. Furthermore σ restricts to an isomorphism

$$\mathbf{d}(G \otimes H) \approx \mathrm{QHom}(\mathbf{A}G, H).$$

PROOF: The fact that σ is an isomorphism follows directly from Proposition 6.39 (with $S = Q$) since $\mathrm{Hom}(G, Q) = \mathrm{Hom}(QG, Q)$ and $\mathrm{Hom}(\mathbf{A}G, QH) = \mathrm{Hom}(Q\mathbf{A}G, QH) = \mathrm{Hom}(\mathrm{Hom}(QG, Q),\, QH)$. Likewise there is an R-linear isomorphism $\sigma_R\colon QR \otimes G \otimes H \xrightarrow{\sigma_R\ \approx} \mathrm{Hom}_R(\mathrm{Hom}_R(R \otimes G, QR),\, QR \otimes H)$, such that σ is essentially just the restriction of σ_R.

By Proposition 1.*, $G \otimes H \ \lhd\ R \otimes G \otimes H$, so that

$$\mathbf{d}(G \otimes H) = (QG \otimes H) \cap \mathbf{d}(R \otimes G \otimes H).$$

Now there is a commutative diagram

$$\begin{tikzcd}
QG \otimes H \arrow[r, "\sigma"] \arrow[d, "\subseteq"] & \mathrm{Hom}(\mathrm{Hom}(G, Q),\, QH) \arrow[d] \\
QR \otimes G \otimes H \arrow[r, "\sigma_R\ \approx"] \arrow[d, equal] & \mathrm{Hom}_R(\mathrm{Hom}_R(R \otimes G, QR),\, QR \otimes H) \arrow[d, "\approx"] \\
QR \otimes G \otimes H \arrow[r, "\tilde{\sigma}_R\ \approx"] \arrow[d] & \mathrm{Hom}_R(\mathrm{Hom}(G, QR),\, QR \otimes H\,) \arrow[d] \\
Q(R \star G) \otimes H \arrow[r, "\bar{\sigma}_R"] & \mathrm{Hom}_R(\mathrm{Hom}(G, R),\, QR \otimes H\,),
\end{tikzcd}$$

where for $\psi \in \mathrm{Hom}(G, R)$, $\bar{\sigma}_R(r \star g \otimes h)(\psi) = \tilde{\sigma}_R(r \otimes g \otimes h)(\psi) = r\psi(g) \otimes h$ and we have used the isomorphism $R \otimes \mathrm{Hom}(G, Q) \approx \mathrm{Hom}(G,\, QR)$ from Lemma 6.40. Note that the composition of the vertical maps on the right hand side of the diagram is the map $\mu\colon \mathrm{QHom}(Q\mathbf{A}G, QH) \to \mathrm{Hom}_R(\mathrm{Hom}(G, R),\, QR \otimes H)$ given in Proposition 6.43. To see that $\bar{\sigma}_R$ is well defined and an isomorphism, observe that it is essentially the composition of the chain of natural isomorphisms

$$\begin{aligned}
(R \star G) \otimes H &\approx (R \star G) \otimes_R (R \otimes H) \\
&\approx \mathrm{Hom}_R(\mathrm{Hom}_R(R \star G, R),\, R \otimes H\,) \\
&\approx \mathrm{Hom}_R(\mathrm{Hom}(G, R),\, R \otimes H\,).
\end{aligned}$$

The second of these isomorphisms is given by Lemma 6.39 since G is R-split and therefore $R \star G$ is a finitely generated projective R-module. This shows that $\bar{\sigma}_R$ restricts to an isomorphism $R \star G \otimes H \approx \mathrm{Hom}_R(\mathrm{Hom}(G, R), R \otimes H)$, so that in particular it maps $\mathbf{d}(R \star G \otimes H)$ isomorphically onto $\mathbf{d}(\mathrm{Hom}_R(\mathrm{Hom}(G, R),\, R \otimes H))$.

Now let $\lambda\colon QG \otimes H \to Q(R \star G) \otimes H$ be the composition of the vertical maps on the left hand side of the diagram. Since the kernel of the map $R \otimes G \otimes H \to (R \star G) \otimes H$ is divisible, we see that

$$\begin{aligned}\mathbf{d}(G \otimes H) &= (QG \otimes H) \cap \mathbf{d}(R \otimes G \otimes H)\\ &= \lambda^{-1}(\mathbf{d}(R \star G \otimes H))\\ &= (\bar{\sigma}_R \lambda)^{-1}(\mathbf{d}(\mathrm{Hom}_R(\mathrm{Hom}(G, R),\ R \otimes H\ )))\end{aligned}$$

so that

$$\begin{aligned}\sigma(\mathbf{d}(G \otimes H)) &= \mu^{-1}(\mathbf{d}(\mathrm{Hom}_R(\mathrm{Hom}(G, R),\ R \otimes H)))\\ &= \mathrm{QHom}(\mathbf{A}G, H)\end{aligned}$$

by Proposition 6.43. Thus σ maps $\mathbf{d}(G \otimes H)$ isomorphically onto $\mathrm{QHom}(\mathbf{A}G, H)$. $\boxed{\checkmark}$

Just as in Chapter 5 we saw that a Butler module can be characterized up to quasi-equality by a Q-vector space, namely QG, and a family of subspaces, namely $QG(\mathbf{t})$ for a certain set of types $\mathbf{t}$, we have now seen that a module G over a discrete valuation ring is determined up to quasi-equality by QG and a K-subspace of $K \otimes QG$, where K is some extension field of Q. One can always choose the quotient field of the completion $\hat{W}$ for K, but in most cases it is possible and often more useful to choose a smaller field.