9. LOCALLY FREE MODULES

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In the earlier chapter your author has tried to promote the point of view that the category of finite rank torsion free modules *under quasi-homomorphisms* is *the* appropriate environment for studying torsion free modules. For locally free modules, however, it turns out that quasi-homomorphisms do not play a very major role.

DEFINITION 9.1. G is **locally free** if G_p is a free W_p-module for all primes p. (Recall from Proposition 0.* that G_p is free over $W_p \iff G_p$ is projective over $W_p \iff G_p$ is finitely generated over W_p.)

PROPOSITION 9.2. (1) *Submodules and homomorphic images of locally free modules are locally free.*

(2) *A rank-one module A is locally free if and only if $\mathbf{t}(A)$ is locally trivial.*
(3) *If G is locally free then $\mathbf{IT}(G)$, $\mathbf{OT}(G)$, and $\mathbf{ET}(G)$ are locally trivial.*
(4) *G is locally free if and only if G is $\mathbf{t}$-bounded for some locally trivial type $\mathbf{t}$.*
(5) *If H is locally free and H dominates G, then G is locally free.*
(6) *If G and H are locally free then so is $G \otimes H$.*
(7) *If H is locally free then so is* $\operatorname{Hom}(G, H)$.
(8) *If W is semi-local then locally free modules are free.*

PROOF: (1) If G_p is a free W_p-module then any submodule or homomorphic image of G_p is a free W_p-module.

(2) By Proposition 2.*.

(3) By Proposition 2.* $\mathbf{IT}(G) \le \mathbf{OT}(G) \le \mathbf{ET}(G)$. It follows immediately that if $\mathbf{ET}(G)$ is locally trivial then so are the other two. But $\mathbf{ET}(G) = \mathbf{t}\left(\bigwedge^r G\right)$, and $\bigwedge^r G$ is a homomorphic image of $G \otimes \cdots \otimes G$ (r copies), hence is locally free by (6). Thus $\mathbf{ET}(G)$ is locally trivial by (2).

(4) ($\Rightarrow$): G is $\mathbf{t}$-bounded, where $\mathbf{t} = \mathbf{OT}(G)$, and by (3) if G is locally free then $\mathbf{OT}(G)$ is locally trivial.

($\Leftarrow$): If $\mathbf{t} = \mathbf{t}(A)$ is locally trivial and G is $\mathbf{t}$-bounded then by (2) A is locally free and by Proposition 4.* A dominates G. Thus by (3) G is locally free.

(5) If H dominates G then by Proposition 4.* H is a submodule of a direct sum of copies of G.

(6) If G_p and H_p are free W_p-modules then so is $G_p \otimes H_p$.

7) If H_p is a free W_p-module then so is $\mathrm{Hom}(G_p, H_p)$.

8) If G is locally free then by 5) $\mathbf{OT}(G)$ is locally trivial. If W is semi-local then it follows from Proposition 2.* that $\mathbf{OT}(G) = \mathbf{t}(W)$. Then G is $\mathbf{t}(W)$-bounded so by Proposition 2.* G is projective. By Proposition 0.* G is free. ☑

PROPOSITION 9.3. *A finite rank torsion free module G is locally free if and only if* $(\forall p)$ p-rank $G = \mathrm{rank}\, G$.

PROOF:

As a special case of Proposition 4.* we have the following:

PROPOSITION 9.4. *Let A be a locally free rank-one module. Let G and H be arbitrary finite rank torsion free modules.*

(1) $G \approx \mathrm{Hom}(A,\, A \otimes G)$.
(2) $\mathrm{Hom}(A \otimes G,\, A \otimes H) \approx \mathrm{Hom}(G, H)$.

PROOF:

Restated, this proposition says the following:

PROPOSITION 9.5. *If A is a rank-one locally free module and $\mathbf{t} = \mathbf{t}(A)$ then the functor $A \otimes$ _ is an equivalence between the category of finite rank torsion free modules under homomorphisms and the category of $\mathbf{t}$-saturated modules. An inverse to this functor is given by* $\mathrm{Hom}(A,$ _ $)$.

PROOF:

LOCALLY FREE MODULES AND QUOTIENT DIVISIBLE MODULES. The class of all finite rank torsion free modules can be thought of as a spectrum, where the quotient divisible modules are at one extreme and the locally free modules are at the other, and all other finite rank modules lie somewhere in between.

PROPOSITION 9.6. (1) *If G is quotient divisible with no non-trivial projective summand and H is locally free, then* $\mathrm{Hom}(G, H) = 0$.

(2) *If G is both quotient divisible and locally free then G is projective.*

PROOF: 2) Let F be an essential free submodule of G. For every prime p, applying Lemma 3.13 to the free W_p-module G_p shows that G_p is quasi-equal to F_p. Since G and F are quotient divisible, Theorem 8.* shows that G and F are quasi-equal. Thus G is projective.

1) If $\varphi \in \mathrm{Hom}(G, H)$ then by Proposition 8.* $\varphi(G)$ is quotient divisible and by Proposition 9.* it is locally free. Therefore by 2) $\varphi(G)$ is projective, hence $\varphi(G)$ is isomorphic to a summand of G. Since G has no non-trivial projective summands, $\varphi(G) = 0$ and therefore $\varphi = 0$. ☑

COROLLARY 9.7. *A locally free semi-prime ring is projective.*

PROOF: By Proposition 7.* a semi-prime ring is quotient divisible. ☑

Proposition 9.6 suggests that in the category of finite rank torsion free modules the classes of quotient divisible modules and of locally free modules play a role somewhat analogous to that of the torsion modules and the torsion free modules in the category of all W-modules. One should not get carried away with this analogy, however, and in particular one should not think that there is a legitimate torsion theory hiding somewhere in the woodwork here.

Recall from Proposition 3.* that if G is a finite rank torsion free module and $\varphi\colon QG \to QH$ then in order for φ to be a quasi-homomorphism from G to H two conditions must be satisfied:

(1) For all primes p, $\varphi \in \mathrm{QHom}(G_p, H_p)$.
(2) For all but finitely many p, $\varphi \in \mathrm{Hom}(G_p, H_p)$.

Recall also from Theorem 8.* that if G is quotient divisible then condition (1) alone is sufficient. On the other hand, if G is locally free we now see that condition (2) alone suffices.

PROPOSITION 9.8. *Let G be a locally free module, let H be any torsion free module, and let $\varphi\colon QG \to QH$. Then $\varphi \in \mathrm{QHom}(G, H)$ if and only if $\varphi \in \mathrm{Hom}(G_p, H_p)$ for all but finitely many primes p.*

PROOF: ($\Rightarrow$): If G_p is a free W_p-module then by Proposition 3.* $\varphi \in \mathrm{QHom}(G_p, H_p)$. If G is locally free then this is true for all primes p so by Proposition 3.* $\varphi \in \mathrm{QHom}(G, H)$ for all but finitely many p. ☑

In parallel to the definition of a quotient divisible module we have the following characterization of locally free modules in terms of their Richman type:

PROPOSITION 9.9. *Let F be an essential projective submodule of G. Then G is locally free if and only if G/F is p reduced.*

PROOF: ($\Rightarrow$): If G is locally free then for all p, G_p/F_p is a finitely generated W_p-module, thus is reduced. Since G/F is torsion, by Proposition 0.* $G/F \approx \bigoplus G_p/F_p$. Thus G/F is reduced.

($\Leftarrow$): If G_p/F_p is reduced then by Proposition 0.* it has finite length, so that G_p is quasi-equal to F_p and hence is a projective W_p-module. ☑

PROPOSITION 9.10. *Let G be a finite rank torsion free module and let G_0 be the quotient divisible core of G. Then there exists a locally free submodule G_ℓ of G such that $G = G_0 + G_\ell$ and $G_0 \cap G_\ell$ is projective.*

PROOF: Let F be a projective submodule of G such that $F \subseteq G_0$ and $G_0/F = \mathbf{d}(G/F)$ (see Proposition 8.*). Then G_0/F is a summand of G/F. Choose $G_\ell \subseteq G$ such that $F \subseteq G_\ell$ and $G/F = G_0/F \oplus G_\ell/F$. Then $G = G_0 + G_\ell$ and $G_0 \cap G_\ell = F$. Furthermore G_ℓ/F is a reduced torsion module, hence is p-reduced for all p. Hence by Proposition 9.9 G_ℓ is locally free. ☑

Unfortunately, one does not get any uniqueness properties for G_ℓ.

PROPOSITION 9.11. *Let G be locally free and $N = \operatorname{nil\,rad}\operatorname{End} G$. Then N is a summand of $\operatorname{End} G$ and $\operatorname{End} G/N$ is projective.*

PROOF: It suffices to prove that $\operatorname{End} G/N$ is projective. In fact, it is a semi-prime ring, hence by Proposition 8.* is quotient divisible. By Proposition 9.* it is also locally free. Therefore it is projective. ☑

PROPOSITION 9.12. *Let G be a locally free module and H a module with no non-trivial projective summand. Let $N = \operatorname{nil\,rad}\operatorname{End} G$. Let $\mu\colon G \otimes H \to G$. Then $\mu(G \otimes H) \subseteq NG$.*

PROOF: Since $\operatorname{Hom}(G \otimes H, G) \approx \operatorname{Hom}(H, \operatorname{Hom}(G, G)) = \operatorname{Hom}(H, \operatorname{End} G)$, μ corresponds to a map $\tilde{\mu}\colon H \to \operatorname{End} G$. Composing this with the quotient map $\operatorname{End} G \to \operatorname{End} G/N$ yields $\bar{\mu}\colon H \to \operatorname{End} G/N$. Since by Proposition 9.11 $\operatorname{End} G/N$ is projective, the image of $\bar{\mu}$ is projective and hence must be isomorphic to a summand of H. Since H has no non-trivial projective summand, $\bar{\mu} = 0$ so that $\tilde{\mu}(H) \subseteq N$. This says that for $g, \in G$, $h \in H$, $\mu(g \otimes h) = \tilde{\mu}(h)(g) \in NG$. ☑

WARFIELD DUALITY.

DEFINITION 9.13. Let A be a locally free rank-one module. Then for any finite rank torsion free G we define the **Warfield dual** of G with respect to A to be $A^*(G) = \operatorname{Hom}(G, A)$. We also write $A^{**}(G) = A^*(A^*(G))$ and let $\varphi\colon G \to A^{**}(G)$ be the natural map defined by $\rho(g)(\varphi) = \varphi(g)$.

THEOREM 9.14. *Let A be a locally free rank-one module and $\mathbf{t} = \mathbf{t}(A)$. Let G and H be any finite rank torsion free modules and let $\sigma\colon A^*(G) \otimes G \to A$ be defined by $\sigma(\gamma \otimes g) = \gamma(g)$.* THEN

(1) *$A^*(G)$ is $\mathbf{t}$-bounded.*
(2) *$A^*(G \otimes H) \approx \operatorname{Hom}(G, A^*(H)) \approx \operatorname{Hom}(H, A^*(G))$.*
(3) *If G is $\mathbf{t}$-bounded then for any H the map $\theta\colon A^*(G) \otimes H \to \operatorname{Hom}(G, A \otimes H)$ given by $\theta(\gamma \otimes h)(g) = \gamma(g)h$ is an isomorphism.*
(4) *The natural map $\rho\colon G \to A^{**}(G)$ is surjective.*
(5) $\operatorname{Ker}\rho = G[\mathbf{t}]$.
(6) *If G is $\mathbf{t}$-bounded and strongly indecomposable then $A^*(G)$ is strongly indecomposable.*
(7) *If G is $\mathbf{t}$-bounded then $\mathbf{OT}(A^*(G)) = [\mathbf{t}\colon \mathbf{IT}(G)]$.*

PROOF: (1) By Proposition 4.* since A is $\mathbf{t}$-bounded then $A^*(G) = \operatorname{Hom}(G, A)$ is $\mathbf{t}$-bounded.

(2) $A^*(G \otimes H) = \operatorname{Hom}(G \otimes H, A) \approx \operatorname{Hom}(G, \operatorname{Hom}(H, A)) = \operatorname{Hom}(G, A^*(H))$ by Proposition 0.*.

(3) If G is $\mathbf{t}$-bounded then by Proposition 4.* A dominates G so the result follows from Proposition 4.*.

(4) Since ρ factors through $G/G[\mathbf{t}]$ it suffices to deal with the case where G is $\mathbf{t}$-bounded, hence dominated by A. It suffices to prove that in this case for each prime p the localized map $G_p \to A^{**}(G)_p$ is an isomorphism. But since A dominates G, by Lemma 4.* $A^{**}(G)_p = \operatorname{Hom}(\operatorname{Hom}(G, A), A)_p \approx \operatorname{Hom}(\operatorname{Hom}(G_p, A_p), A_p)$. Since G_p is a free W_p-module and $A_p \approx W_p$, then result is then standard and easy.

(5) Clear from (4) because a quasi-direct decomposition of $A^*(G)$ induces a quasi-direct decomposition of $A^{**}(G)$, and hence of G.

(6) The rank-one homomorphic images of $A^*(G)$ are of the form $A^*(B)$, where B is a pure submodule of G, and conversely. Thus $\mathbf{OT}(A^*(G))$ is the least upper bound of the set of types $\mathbf{t}(A^*(B))$ with $\mathbf{t}(B) \in \mathbf{T}(G)$. But $\mathbf{t}(A^*(B)) = \mathbf{t}(\operatorname{Hom}(B, A) = [\mathbf{t}\colon \mathbf{t}(B)]$. ☑

PROPOSITION 9.15. *If* $0 \to H \to G \to K \to 0$ *is a short exact sequence of* $\mathbf{t}(A)$*-bounded modules, then* $0 \to A^*(K) \to A^*(G) \to A^*(H) \to 0$ *is exact.*

PROOF: This is just Proposition 4.*. ☑

PROPOSITION 9.16. *Let* $\Gamma\colon \operatorname{Hom}(H, G) \to A^*(A^*(G) \otimes H)$

be given by

$$\Gamma(\psi) = \sigma(1 \otimes \psi)\colon A^*(G) \otimes H \to A.$$

If G *is* $\mathbf{t}$*-bounded then* Γ *is an isomorphism.*

PROOF: Let $\gamma \in A^*(G)$, $h \in H$, and $\psi \in \operatorname{Hom}(H, G)$. Then $\Gamma(\psi)(\gamma \otimes h) = \sigma(\gamma \otimes \psi(h)) = \rho(\psi(h))(\gamma)$, so that Γ is the composition

$$\operatorname{Hom}(H, G) \to \operatorname{Hom}(H, A^{**}(G)) \to A^*(A^*(G) \otimes H),$$

where the first of these maps is induced by ρ and the second is the isomorphism given in (2). Hence if G is $\mathbf{t}$-bounded then Γ is an isomorphism. ☑

PROPOSITION 9.17. *Let* G *be locally free and let* $A = \bigwedge^r G$*, where* $r = \operatorname{rank} G$*. Then* $A^*(G) \approx \bigwedge^{r-1} G$.

PROOF: Any $g_1 \wedge \cdots \wedge g_{r-1} \in \bigwedge^{r-1} G$ yields a map $\alpha\colon G \to A$ by $\alpha(x) = g_1 \wedge \cdots \wedge g_{r-1} \wedge x \in \bigwedge^r G = A$. This gives a map $\bigwedge^{r-1} G \to A^*(G)$. Furthermore if $g_1 \wedge \cdots \wedge g_{r-1} \neq 0$ then $\alpha \neq 0$ since $g_1 \wedge \cdots \wedge g_{r-1} \wedge x \neq 0$ for any x not belonging to the pure submodule of G generated by $g_1, \ldots, g_r$. Thus the map $\bigwedge^{r-1} G \to A^*(G)$ is monic. The hard part is to see that it is surjective.

Let $\alpha' \in A^*(G)$ and let $K = \operatorname{Ker} \alpha'$. Let $g_1, \ldots, g_{r-1}$ be a linearly independant set in K and fix $g' \in G$ with $g' \notin K$. Then $g_1 \wedge \cdots \wedge g_{r-1} \wedge g' \neq 0 \in A$ because $g_1, \ldots, g_{r-1}, g'$ are linearly independant. Since $\operatorname{rank} A = 1$ there exists $u \in Q$ with $\alpha'(g') = ug_1 \wedge \cdots \wedge g_{r-1} \wedge g'$. Then the map $\alpha\colon x \mapsto ug_1 \wedge \cdots \wedge g_{r-1} \wedge x$ is an element of $QA^*(G)$ and $\operatorname{Ker} \alpha = K = \operatorname{Ker} \alpha'$. It follows that α is a multiple of α'. But since $\alpha(g') = \alpha'(g')$, it follows that $\alpha = \alpha'$.

Since we are given only that $u \in Q$, it remains to be shown that $ug_1 \wedge \cdots \wedge g_{r-1} \in \bigwedge^{r-1} G$. It will sufffice to prove that $ug_1 \wedge \cdots \wedge g_{r-1} \in \bigwedge^{r-1} G_p$ for every p. By hypothesis G_p is a free W_p-module. Choose a basis $k_1, \ldots, k_{r-1}, g_r$ for G_p with $k_1, \ldots, k_{r-1} \in K$. Then $k_1 \wedge \cdots \wedge k_{r-1}$ is a generator for the cyclic module $\bigwedge K_p^{r-1}$. Thus $g_1 \wedge \cdots \wedge g_{r-1} = w' k_1 \wedge \cdots \wedge k_{r-1}$ for some $w' \in W$. Then $ug_1 \wedge \cdots \wedge g_{r-1} \wedge g_r = uw' k_1 \wedge \cdots \wedge k_{r-1} \wedge g_r \in \bigwedge^r G_p$. Since $k_1, \ldots, k_{r-1}, g_r$ is a basis for $\bigwedge^r G_p$ it follows that $uw' \in W$ so that $ug_1 \wedge \cdots \wedge g_{r-1} = uw' k_1 \wedge \cdots \wedge k_{r-1} \in \bigwedge^{r-1} G_p$. Since this is true for every prime p it follows that $ug_1 \wedge \cdots \wedge g_{r-1} \in \bigwedge^{r-1} G$. ☑

PROPOSITION 9.18. *Let H be $\mathbf{t}(A)$-bounded and $\varphi\colon G \to H$. Then φ is a monomorphism mapping G onto an essential submodule of H if and only if $A^*(\varphi)$ is a monomorphism mapping $A^*(H)$ onto an essential submodule of $A^*(G)$.*

PROOF: ($\Rightarrow$): We may assume that $G \subseteq H$ and φ is the inclusion. Thus if $\gamma \in A^*(H)$ then $A^*(\varphi)(\gamma)$ is simply the restriction of γ to G. Thus since G is essential in H, if $\gamma \neq 0$ then $A^*(\varphi)(\gamma) \neq 0$. Thus $A^*(\varphi)$ is monic. Now let $\gamma \in A^*(G)$. Then $\mathbf{t}(\gamma(H)) \leq \mathbf{OT}(H) \leq \mathbf{t}(A)$, so by Proposition 2.* for some $w \neq 0 \in W$, $w\gamma(H) \subseteq A$. Thus $w\gamma \in A^*(H)$. Thus every element of $A^*(G)$ has a non-trivial multiple in $A^*(H)$, so $A^*(G)$ is an essential submodule of $A^*(H)$.

($\Leftarrow$): If $A^*\varphi$ is a monomorphism mapping $A^*(H)$ onto an essential submodule of $A^*(G)$, then by the previous paragraph $A^{**}(\varphi)$ is a monomorphism mapping $A^{**}(G)$ onto an essential submodule of $A^{**}(H)$. Using the natural isomorphism ρ between A^{**} and the identity functor, we conclude that φ is a monomorphism mapping G onto a pure submodule of H. ☑

PROPOSITION 9.19. *Let G_1 and G_2 be $\mathbf{t}$-bounded modules with the same divisible hull. Then $A^*(G_1 + G_2) = A^*(G_1) \cap A^*(G_2)$ and $A^*(G_1 \cap G_2) = A^*(G_1) + A^*(G_2)$.*

PROOF: Because G_1 and G_2 are $\mathbf{t}$-bounded, $A^*(G_1)$ and $A^*(G_2)$ are essential submodules of $\mathrm{Hom}(QG_1, Q) = \mathrm{Hom}(QG_2, Q)$. Now if $\varphi\colon QG_1 \to Q$ then $\varphi(G_1 + G_2) \subseteq A \iff \varphi(G_1) \subseteq A$ & $\varphi(G_2) \subseteq A$. Thus $A^*(G_1 + G_2) = A^*(G_1 \cap A^*(G_2)$. And from this we derive $A^*(G_1 \cap G_2) = A^*(A^{**}(G_1) \cap A^{**}(G_2)) = A^*(A^*(A^*(G_1) + A^*(G_2)) = A^*(G_1) + A^*(G_2)$. ☑

Stated another way, Proposition 9.19 says that the functor A^* affords a lattice anti-isomorphism between the set of $\mathbf{t}$-bounded essential submodules of any finite dimensional Q-space V and the set of $\mathbf{t}$-bounded essential submodules of $\mathrm{Hom}(V, Q)$. Combining this with any isomorphism from $\mathrm{Hom}(V, Q)$ to V will give a lattice anti-isomorphism from the lattice of $\mathbf{t}$-bounded essential submodules of V onto itself.

The following corollary may be compared to Proposition 8.*. There seems to be a hint here that $\mathbf{ET}(G)$ may play the same role for locally free modules that the family of numbers $\{\text{p-rank}\, G\}_{p \in \mathrm{Spec}\, W}$ plays for quotient divisible modules.

COROLLARY 9.20. *If G is an essential submodule of a locally free module H and $\mathbf{ET}(G) = \mathbf{ET}(H)$, then G is quasi-equal to H.*

PROOF: Let $A = \bigwedge^r G$ and $\mathbf{t} = \mathbf{t}(A) = \mathbf{ET}(G) = \mathbf{ET}(H)$, so that $A \sim \bigwedge^r H$. By Proposition 9.19 $\bigwedge^{r-1} G \approx A^*(G)$ and $\bigwedge^{r-1} H \sim A^*(H)$. Now the inclusion map $G \hookrightarrow H$ induces a monomorphism $\bigwedge^r G \to \bigwedge^r H$, and thus a monomorphism $A^*(G) \to A^*(H)$. On the other hand, by Proposition 9.19, applying the functor A^* to the inclusion yields a monomorphism $A^*(H) \to A^*(G)$. It thus follows that $A^*(G)$ and $A^*(H)$ are quasi-isomorphic. Thus G and H are quasi-isomorphic, and since $G \subseteq H$ they must be quasi-equal. ☑

PROPOSITION 9.21. *Let A be a locally free rank-one module, $\mathbf{t} = \mathbf{t}(A)$, and let G be a $\mathbf{t}$-bounded module. Let $\delta\colon A \to \operatorname{Hom}(G, A \otimes G)$ be defined by $\delta(a)(g) = a \otimes g$ and let $\theta\colon A^*(G) \otimes G \to \operatorname{Hom}(G, A \otimes G)$ be defined as in Proposition 9.2. Then there is an isomorphism*

$$\Delta\colon \operatorname{Hom}(G, H) \approx \operatorname{Hom}(A, A^*(G) \otimes H)$$

given by $\Delta(\varphi) = (1 \otimes \varphi)\theta^{-1}\delta$.

PROOF: By the naturality of θ, $\Delta(\varphi) = \theta^{-1}(1 \otimes \varphi)_*\delta$ and so Δ is simply the composition $\operatorname{Hom}(G, H) \approx \operatorname{Hom}(A \otimes G, A \otimes H) \approx \operatorname{Hom}(A, \operatorname{Hom}(G, A \otimes H)) \approx \operatorname{Hom}(A, A^*(G) \otimes H)$. ☑

There is an intriguing parallel here to Theorem 8.* for Arnold duality. Theorem 8.* implies that if G is quotient divisible then $\operatorname{QHom}(G, H) \approx \operatorname{Hom}(Q, \mathbf{A}(G) \otimes H)$.

COROLLARY 9.22. *If G is $\mathbf{t}$-bounded then $A \otimes \operatorname{Hom}(G, H) \approx (A^*(G) \otimes H)(\mathbf{t})$.*

PROOF: By Proposition 9.21 and Proposition 4.* $A \otimes \operatorname{Hom}(G, H) \approx A \otimes \operatorname{Hom}(A, A^*(G) \otimes H) \approx (A^*(G) \otimes H)(\mathbf{t})$. ☑

COROLLARY 9.23. *If G is $\mathbf{t}$-bounded then $(G \otimes H)(\mathbf{t}) \approx A \otimes \operatorname{Hom}(A^*(G), H)$.*

PROOF: Since G is $\mathbf{t}$-bounded, $(G \otimes H)(\mathbf{t}) \approx (A^{**}(G) \otimes H)(\mathbf{t}) \approx A \otimes \operatorname{Hom}(A^*(G), H)$ by Proposition 9.22. ☑

MODULES WITH NON-TRIVIAL TRACE.

LEMMA 9.24. *Let G be $\mathbf{t}$-bounded and let $\delta\colon A \to \operatorname{Hom}(G, A \otimes G)$, $\sigma\colon A^*(G) \otimes G \to A$, and $\theta\colon A^*(G) \otimes G \to \operatorname{Hom}(G, A \otimes G)$ be defined as in Theorem 9.14 and Proposition 9.21. Let $\varphi \in \operatorname{End} G$. Then $\operatorname{Trace}\varphi = \sigma(1 \otimes \varphi)\theta\delta(1) \in W$.*

PROOF: Since A dominates G, Proposition 1.* shows that it suffices to prove the result locally. Hence no generality is lost in supposing G free and $A = W$. Let $g_1, \ldots, g_r$ be a basis for G and let $\gamma_1, \ldots, \gamma_r \in \operatorname{Hom}(G, Q)$ be the dual basis. Then in fact $\gamma_i \in A^*(G)$ for all i since $\gamma_i(G) \subseteq W = A$. Now $\theta^{-1}\delta(1) = \sum \gamma_i \otimes g_i$ and for $\varphi \in \operatorname{End} G$, $\sigma(1 \otimes \varphi)\theta^{-1}\delta(1) = \sum \gamma_i(\varphi(g_i)) = \operatorname{Trace}\varphi$. (In fact $\gamma_i(\varphi(g_i))$ is simply the i^{th} diagonal entry in the matrix for φ.) Furthermore since $\sigma(1 \otimes \varphi)\theta^{-1}\delta \in \operatorname{End} A \approx W$, then $\operatorname{Trace}\varphi = \sigma(1 \otimes \varphi)\theta^{-1}\delta(1) \in W$. ☑

We say that G **has non-trivial trace** if there exists $\varphi \in \operatorname{End} G$ such that $\operatorname{Trace}\varphi \neq 0$. It is well known, of course, that if $\varphi = 1 \in \operatorname{End} G$ then $\operatorname{Trace}\varphi = \operatorname{rank} G \in W$. Thus if $\operatorname{char} W \neq 0$ then every module has non-trivial trace.

PROPOSITION 9.25. *Let G be $\mathbf{t}$-bounded. The following conditions are equivalent:*

1. *G has non-trivial trace.*
2. *$\sigma\colon A^*(G) \otimes G \to A$ is a quasi-split surjection.*
3. *$\delta\colon A \to \operatorname{Hom}(G, A \otimes G)$ is a quasi-split monomorphism.*

PROOF: 1) $\Rightarrow$ 2) & 3): By Lemma 9.24 if $\operatorname{Trace}\varphi \neq 0$ then $\sigma(1\otimes\varphi)\theta^{-1}\delta \neq 0 \in \operatorname{End} A$. Since all non-trivial endomorphisms of A are quasi-automorphisms (because $\operatorname{QEnd} A = Q$), it follows that δ and σ split.

(2) $\Rightarrow$ (1): Let $\eta \in \operatorname{QHom}(A, A^*(G)\otimes G)$ be a quasi-splitting for σ. Then for some $w \neq 0 \in W$, $w\eta \in \operatorname{Hom}(A, A^*(G)\otimes G)$. By Proposition 9.21 $w\eta = (1\otimes\varphi)\theta^{-1}\delta$ for some $\varphi \in \operatorname{End} G$. Then by Lemma 9.24 $\operatorname{Trace}\varphi = w\sigma\eta(1) = w \neq 0$.

(3) $\Rightarrow$ (1): Analogous. ☑

PROPOSITION 9.26. *Let G be a $\mathbf{t}$-bounded strongly indecomposable module. Let $N = \operatorname{nil\,rad}\operatorname{End} G$ and $Z = \operatorname{Center}(\operatorname{End} G/N)$. Then G has non-trivial trace if and only if QZ is separable over Q and $(\operatorname{rank} G)/[QZ\colon Q]$ is not a multiple of $\operatorname{char} W$.*

PROOF: Let $D = \operatorname{QEnd} G/N$. Since G is strongly indecomposable, D is a skew field by Proposition 3.*. Let $X_i = N^iQG/N^{i+1}QG$ for $i = 1, 2, \dots, m$, where $N^m = 0$. Let $X = \bigoplus X_i$. Then X is a D-module and is canonically isomorphic to QG as a Q-vector space. For $\varphi \operatorname{QEnd} G$ let φ' be the image of φ in D. Since X is a D-space, left multiplication by φ' yields a scalar linear transformation $\hat\varphi$ on X. Furthermore if we identify X and QG by using the obvious map, then φ also becomes a linear transformation on X and $\varphi - \hat\varphi$ is nilpotent, so $\operatorname{Trace}\varphi = \operatorname{Trace}\hat\varphi$. Then $\operatorname{Trace}\hat\varphi = (\dim_D X)\operatorname{Trace}_{D/Q}\varphi' = (\dim_X D)\operatorname{Trace}_{QZ/Q}(\operatorname{Trace}_{D/QZ}\varphi')$, where by $\operatorname{Trace}_{D/Q}\varphi'$ is meant, of course, the usual field-theoretic trace of $\varphi' \in D$.

($\Leftarrow$): If QZ is separable over Q then there exists $\varphi' \in QZ \subseteq D$ with $\operatorname{Trace}_{QZ/Q}\varphi' \neq 0$. If furthermore $\dim_{QZ} X = (\operatorname{rank} G)/[Z\colon Q]$ is not a multiple of $\operatorname{char} W$, then $\operatorname{Trace}_X\hat\varphi = (\dim_{QZ} X)\operatorname{Trace}_{QZ/Q}\varphi' \neq 0$. Thus $\operatorname{Trace}\varphi \neq 0$ for any pre-image $\varphi \in \operatorname{QEnd} G$ for φ'.

($\Rightarrow$): Now if QZ is not separable over Q then it is known from field theory that $\operatorname{Trace}_{QZ/Q}\varphi' = 0$ for all φ'. And if $\dim_D X$ is a multiple of $\operatorname{char} W$ then clearly $\operatorname{Trace}\hat\varphi = 0$ for all scalar linear transformations $\hat\varphi$ on the D-space X. And finally if $[D\colon QZ]$ is a multiple of $\operatorname{char} W$ it is known from the theory of simple algebras [Bourbaki] that $\operatorname{Trace}_{D/QZ}\varphi' = 0$ for all $\varphi' \in D$. Thus $\operatorname{Trace}\varphi = \operatorname{Trace}\hat\varphi = 0$ for all φ if either QZ is not separable over Q or $(\operatorname{rank} G)/[QZ\colon Q] = [D\colon QZ]\dim_D X$ is a multiple of $\operatorname{char} W$. ☑

COROLLARY 9.27. *Let G be locally free with non-trivial trace.*

(1) *For any H the natural map $\delta_H\colon H \to \operatorname{Hom}(G, H \otimes G)$ given by $\delta_H(h)(g) = h \otimes g$ is a quasi-split monomorphism and the adjointness isomorphism $\operatorname{Hom}(X \otimes G, H) \approx \operatorname{Hom}(X, \operatorname{Hom}(G,H)$ takes quasi-split monomorphisms to quasi-split monomorphisms.*

(2) *If H dominates G then the natural map $\sigma\colon \operatorname{Hom}(G,H) \otimes G \to H$ given by $\sigma_H(\psi \otimes g) = \psi(g)$ is a quasi-split surjection and the adjointness isomorphism $\operatorname{Hom}(X, \operatorname{Hom}(G,H)) \approx \operatorname{Hom}(X \otimes G, H)$ takes split quasi-surjections to split quasi-surjections.*

PROOF: Choose A so that $\mathbf{t}(A) = \mathbf{OT}(G)$. First we show that δ_H and σ_H are quasi-split. By Lemma 9.4 to see that δ_H is quasi-split monic it suffices to show that $A \otimes \delta_H\colon A \otimes H \to A \otimes \operatorname{Hom}(G, H \otimes G)$ is quasi-split monic, and for this it suffices to show that the composition

$$\delta_{A\otimes H}\colon A \otimes H \to A \otimes \operatorname{Hom}(G, H \otimes G) \to \operatorname{Hom}(G, A \otimes H \otimes G)$$

is quasi-split monic. Hence, replacing H with $A \otimes H$ we may assume WLOG that H dominates A and G.

With this assumption, the map $\theta_H\colon \operatorname{Hom}(G,H) \otimes G \to \operatorname{Hom}(G, G \otimes H)$ given by $\theta_H(\varphi \otimes g)(g') = g \otimes \varphi(g')$ is an isomorphism (c.f. Lemma 4.*) and if $h \in H$ then there exists $\eta \in \operatorname{QHom}(A,H)$ with $\eta(1) = h$. Now apply the naturality of δ, σ, and θ to see that if $\varphi \in \operatorname{End} G$ then

$$\begin{aligned}\sigma_H(1 \otimes \varphi)\theta_H^{-1}\delta_H(h) &= \sigma_H(1 \otimes \varphi)\theta_H^{-1}\delta_H\eta(1)\\ &= \eta\sigma_A(1 \otimes \varphi)\theta_A^{-1}\delta_A(1) = \eta(\operatorname{Trace}\varphi) = (\operatorname{Trace}\varphi)h.\end{aligned}$$

Thus clearly if there exists $\varphi \in \operatorname{End} G$ with $\operatorname{Trace}\varphi \neq 0$ then

$$\sigma_H(1 \otimes \varphi)\theta_H^{-1}\delta_H = \operatorname{Trace}\varphi \neq 0 \in W \subseteq \operatorname{End} H$$

and since multiplication by $\operatorname{Trace}\varphi$ is a quasi-automorphism in $\operatorname{End} H$ it follows that δ_H and σ_H are quasi-split as required.

Now if $\varphi \in \operatorname{Hom}(X \otimes G, H)$ then its adjoint in $\operatorname{Hom}(X, \operatorname{Hom}(G,H))$ can be factored as

$$\varphi_*\delta_X\colon X \to \operatorname{Hom}(G, X \otimes G) \to \operatorname{Hom}(G,H).$$

Since δ_X is quasi-split monic, if φ is quasi-split monic then so is its adjoint. Likewise the adjoint for $\psi \in \operatorname{Hom}(X, \operatorname{Hom}(G,H))$ factors as $\sigma_H(\psi\otimes 1)\colon X\otimes G \to \operatorname{Hom}(G,H)\otimes G \to H$, which is a split quasi-surjection if ψ is. ☑

Whereas examples of failure of cancellation in direct sum decompositions are fairly surprising and need to be somewhat carefully contrived, for tensor products cancellation is not something one even expects. For instance if $G = Q$ then from an isomorphism $G \otimes H \approx G \otimes H'$ we can conclude only that $\operatorname{rank} H = \operatorname{rank} H'$. On the other hand, Proposition 9.4 shows that if A is a locally free rank-one module then we do get cancellation: $A \otimes H \approx A \otimes H'$ implies $H \approx H'$. One of the themes of this chapter is that properties which hold for rank-one locally free modules often are valid in somewhat weakened form for locally free modules in general. Thus one might even be rash enough to conjecture that if G is locally free then $G \otimes H \sim G \otimes H'$ implies $H \sim H'$. Your author will acknowledge that he has invested a considerable amount of time in investigating this unlikely conjecture. In fact the greater part of the material in this chapter was developed pursuant to that investigation. Unfortunately, the closest he has come to success is the following corollary:

COROLLARY 9.28. *If G is locally free with non-trivial trace, then for any finite rank torsion free H there exist, up to quasi-isomorphism, at most finitely many H' such that $G \otimes H \sim G \otimes H'$.*

PROOF: If $G \otimes H \sim G \otimes H'$ then by Corollary 9.27 H' is isomorphic to a quasi-summand of $\operatorname{Hom}(G,\, H' \otimes G) \sim \operatorname{Hom}(G,\, H \otimes G)$. By Jónsson's Theorem (Theorem 3.24) $\operatorname{Hom}(G,\, H \otimes G)$ has, up to quasi-isomorphism, only finitely many quasi-summands. ☑

PROPOSITION 9.29. *Let G be $\mathbf{t}$-bounded and for any H let $\Delta\colon \operatorname{Hom}(G,H) \to \operatorname{Hom}(A,\, A^*(G) \otimes H)$ and $\Gamma\colon \operatorname{Hom}(H,G) \to \operatorname{Hom}(A^*(G) \otimes H,\, A)$ be the isomorphisms from Theorem 9.14 and Proposition 9.21. Let $\varphi \in \operatorname{Hom}(G,H)$ and $\psi \in \operatorname{Hom}(H,G)$.*

1. *If G has non-trivial trace and φ is a quasi-split monomorphism then so is $\Delta(\varphi)$.*
2. *If G has non-trivial trace and ψ is a quasi-split quasi-surjection, then so is $\Gamma(\psi)$.*
3. *If G is strongly indecomposable and $\Delta(\varphi)$ is a quasi-split monomorphism then so is φ.*
4. *If G is strongly indecomposable and $\Gamma(\psi)$ is a quasi-split quasi-surjection then so is ψ.*

PROOF: Since $\Gamma(\psi) = \sigma(1 \otimes \psi)$ and $\Delta(\varphi) = (1 \otimes \varphi)\theta^{-1}\delta$, Lemma 9.24 shows that

$$\Gamma(\psi)\Delta(\varphi) = \operatorname{Trace} \psi\varphi.$$

Now if φ is quasi-split monic then we can choose $\psi \in \operatorname{Hom}(H,G)$ so that $\psi\varphi$ is any desired endomorphism of G. Hence if G has non-trivial trace we can choose ψ so that $\Gamma(\psi)\Delta(\varphi) = \operatorname{Trace} \psi\varphi \neq 0 \in \operatorname{End} A \approx W$, and we conclude that $\Delta(\varphi)$ is quasi-split surjective. Analogously we see that if $\psi \in \operatorname{Hom}(H,G)$ is a split quasi-surjective then $\Gamma(\psi)$ is as well.

Conversely, if $\Delta(\varphi)$ is quasi-split monic then by Proposition 9.* the splitting map has the form $\Gamma(\psi)$ for some $\psi \in \operatorname{QHom}(H,G)$, and so $\operatorname{Trace} \psi\varphi = \Delta(\varphi)\Gamma(\psi) \neq 0$. Since nilpotent linear transformation have trivial trace, this means that $\psi\varphi$ is not nilpotent. Thus if G is strongly indecomposable then by Proposition 3.* $\psi\varphi$ is a quasi-automorphism of G so that φ is also quasi-split monic. Likewise if $\Gamma(\psi)$ is a quasi-split surjection then ψ is as well. ☑

QUASI-SUMMAND OF TENSOR PRODUCTS. In Chapter 5 we saw that if G and H are quotient divisible Butler modules then $G \otimes H$ will often have rank-one quasi-summands. In fact, if one looks at the sequence G, $G \otimes G$, $G \otimes G \otimes G, \ldots$ then eventually a rank-one quasi-summand is inevitable.

For locally free modules the situation is surprisingly different. We have seen in Proposition 9.25 that if G is $\mathbf{t}(A)$-bounded with non-trivial trace then the rank-one module A will be a quasi-summand of $A^*(G) \otimes G$. It now turns out that if G and H strongly indecomposable then this is the only case where $G \otimes H$ can have a rank-one quasi-summand.

THEOREM 9.30. *Let G and H be strongly indecomposable modules. Suppose that $G \otimes H$ has a locally free rank-one quasi-summand A. Let $\mathbf{t} = \mathbf{t}(A)$.* THEN

1. *G and H are locally free and $\mathbf{t}$-bounded and have non-trivial trace.*
2. $H \sim A^*(G)$.
3. $\mathbf{t}(A) = \mathbf{IT}(G)\mathbf{OT}(H)$.
4. *Let $N = \operatorname{nil\,rad} \operatorname{End} G$. Then a maximal completely decomposable quasi-summand of $G \otimes H$ has rank equal to* $\operatorname{rank}(\operatorname{End} G) - \operatorname{rank} N$.

PROOF: Since a split quasi-surjection $G \otimes H \to A$ must vanish on $G[\mathbf{t}] \otimes H$ and by Proposition 9.14 $G/G[\mathbf{t}] \approx A^{**}(G)$, such a split quasi-surjection induces a map $A^{**}(G) \otimes H \to A$ which must also be a split quasi-surjection. By Jónsson's Theorem (Theorem 3.24), A is then a quasi-summand of $A^*(K) \otimes H$ for some strongly indecomposable quasi-summand K of $A^*(G)$. Since K is $\mathbf{t}$-bounded, Proposition 9.29 yields a split quasi-surjection from H to K. Since H is by assumption strongly indecomposable, thus $H \sim K$, so that in particular H is $\mathbf{t}$-bounded and hence by Lemma 9.2 is locally free and by the proof of Proposition 9.29 has non-trivial trace. By symmetry, G is also locally free, $\mathbf{t}$-bounded, and has non-trivial trace. Thus by Proposition 9.14 $A^*(G)$ is strongly indecomposable. Since K is a quasi-summand of $A^*(G)$ we conclude that $H \sim K \sim A^*(G)$.

Furthermore $\mathbf{OT}(H) = \mathbf{OT}(A^*(G)) = [\mathbf{t} \colon \mathbf{IT}(G)]$ and so $\mathbf{IT}(G)\mathbf{OT}(H) = \mathbf{t}$. Finally this shows that a maximal completely decomposable submodule if $G \otimes H$ is $\mathbf{t}$-homogeneous and thus corresponds to a maximal linearly independent set of quasi-split monomorphisms from A to $G \otimes H \sim G \otimes A^*(G)$. Hence by Proposition 9.29 and the fact that G is strongly indecomposable it corresponds to a maximal linearly independent set of quasi-automorphisms of G. Thus the rank of such a maximal completely decomposable quasi-summand equals $\operatorname{rank}(\operatorname{End} G/N) = \operatorname{rank} \operatorname{End} G - \operatorname{rank} N$, where $N = \operatorname{nil\,rad} \operatorname{End} G$. ☑

REMARK 9.31. Call two strongly indecomposable modules G and H **similar** if there exist rank-one locally free modules A_1 and A_2 such that $A_1 \otimes G_1 \sim A_2 \otimes G_2$. If G is locally free then the Warfield dual of G defines a similarity class independent of A, since if $\mathbf{OT}(G) = \mathbf{t}(A_1)$ and $\mathbf{t}(A) \geq \mathbf{t}(A_1)$ then $A^*(G) \approx A^*(A_1) \otimes A_1^*(G)$ and $A^*(A_1)$ is a locally free rank-one module. Thus Theorem 9.17 shows that if G is locally free and strongly indecomposable with non-trivial trace then there exists a strongly indecomposable module H such that $G \otimes H$ has a rank-one quasi-summand, and that

H is unique up to similarity. By Proposition 9.* we can choose $H = \bigwedge^{r-1} G$, where $r = \operatorname{rank} G$.

LEMMA 9.32. *Let G be a strongly indecomposable module and $\eta\colon \operatorname{Hom}(G \otimes H,\, A)$ be a split quasi-surjection. Let $\psi \in \operatorname{Hom}(G,\, A^*(H))$ be adjoint to η. Then G is $\mathbf{t}$-bounded with non-trivial trace and ψ is a split quasi-monomorphism.*

PROOF: Since A is a quasi-summand of $G \otimes H_1$ for some strongly indecomposable quasi-summand H_1 of H, Theorem 9.31 shows that G is $\mathbf{t}$-bounded with non-trivial trace, so that $\rho\colon G \to A^{**}(G)$ is an isomorphism and A is isomorphic to a quasi-summand of $A^{**}(G) \otimes H$. Since $A^*(G)$ is strongly indecomposable Proposition 9.29 implies that the map $\psi_1 \in \operatorname{Hom}(H,\, A^*(G))$ adjoint to η is a split quasi-surjection. Thus $A^*(\psi_1)$ is a split quasi-monomorphism. But for $g \in G$, $h \in H$,

$$(A^*(\psi_1)\rho)(g)(h) = (A^*(\psi_1)(\rho(g)))(h) = \rho(g)(\psi_1(h)) = \psi_1(h)(g) = \eta(g \otimes h) = \psi(g)(h)$$

so that $\psi = A^*(\psi_1)\rho$. Thus ψ is a split quasi-monomorphism. ☑

THEOREM 9.33. *Let K be a locally free module with non-trivial trace. Let G be a strongly indecomposable module and H a module such that K is quasi-isomorphic to a quasi-summand of $G \otimes H$. Then G is likewise locally free with non-trivial trace and is isomorphic to a quasi-summand of $\operatorname{Hom}(H, K)$. More specifically, the adjointness isomorphism $\operatorname{Hom}(G \otimes H,\, K) \approx \operatorname{Hom}(G,\, \operatorname{Hom}(H, K))$ takes split quasi-surjections to split quasi-monomorph isms.*

PROOF: Choose A so that $\mathbf{t}(A) = \mathbf{OT}(K)$. Then Proposition 9.14 yields the following isomorphisms:

$$\begin{aligned}\operatorname{Hom}(G \otimes H,\, K) &\approx A^*(G \otimes H \otimes A^*(K)) \\ &\approx \operatorname{Hom}(G,\, A^*(H \otimes A^*(K)) \approx \operatorname{Hom}(G,\, \operatorname{Hom}(H, K)),\end{aligned}$$

and it is easily seen that the composition of these is the adjointness isomorphism in question. Proposition 9.29 and Lemma 9.32 then show that split quasi-surjections are taken to split quasi-monomorphisms and that G is $\mathbf{t}$-bounded, hence locally free, with non-trivial trace. ☑

COROLLARY 9.34. *If K is locally free with non-trivial trace then for any H there exist up to quasi-isomorphism at most finitely many strongly indecomposable modules G such that K is isomorphic to a quasi-summand of $G \otimes H$. If H has non-trivial trace and K is strongly indecomposable and dominates H, then there is at least one such G.*

PROOF: By Theorem 9.33 the possible strongly indecomposable modules G such that K is isomorphic to a quasi-summand of $G \otimes H$ lie among the quasi-summands of $\operatorname{Hom}(H, K)$. Thus by Jónsson's Theorem there are only finitely many such G.

Now by Corollary 9.27 if H has non-trivial trace and K dominates H then there is a quasi-split map $\operatorname{Hom}(H, K) \otimes H \to K$. Thus by Jónsson's Theorem if K is strongly indecomposable then there exists at least one strongly indecomposable quasi-summand G of $\operatorname{Hom}(H, K)$ such that K is isomorphic to a quasi-summand of $G \otimes H$. ☑

COROLLARY 9.35. *Let G be a quasi-direct sum of strongly indecomposable modules with non-trivial trace. Let H be a locally free module with non-trivial trace dominating G. Then the map $\rho_1\colon G \to \mathrm{Hom}(\mathrm{Hom}(G,H),\,H)$ given by $\rho_1(g)(\varphi) = \varphi(g)$ is a quasi-split monomorphism.*

PROOF: It suffices to prove that theorem for each strongly indecomposable quasi-summand in some quasi-direct decomposition of G, hence no generality is lost in supposing G strongly indecomposable. By Corollary 9.27 $\sigma\colon G \otimes \mathrm{Hom}(G,H) \to H$ is a quasi-split surjection. By Theorem 9.34 the adjoint map $\sigma'\colon G \to \mathrm{Hom}(\mathrm{Hom}(G,H),\,H)$ is a quasi-split monomorphism. But $\sigma' = \rho_1$: in fact, for any $g \in G$, $\varphi \in \mathrm{Hom}(G,H)$, $\sigma'(g)(\varphi) = \varphi(g \otimes \varphi) = \varphi(g) = \rho_1(g)(\varphi)$. ☑

COROLLARY 9.36. *Let G be strongly indecomposable and let $G \otimes H = K \oplus K'$, where $K \neq 0$ and K is locally free with non-trivial trace. Let G_1 be a submodule of G.*

1. *If $K \subseteq G_1 \otimes H$ then G_1 is quasi-equal to G.*
2. *If $G_1 \otimes H \subseteq K'$ then $G_1 = 0$.*

PROOF: 1) Let $K \subseteq G_1 \otimes H$. Then by Proposition 3.* K is a quasi-summand of $G_1 \otimes H$. If we suppose, without loss of generality, that K is strongly indecomposable, then by Jónsson's Theorem K is a quasi-summand of $G_2 \otimes H$ for some strongly indecomposable quasi-summand G_2 of G_1. Furthermore if $\mu\colon G \otimes H \to K$ is a split quasi-surjection then the restriction of μ to $G_2 \otimes H \to K$ is also a quasi-split surjection. Call this restriction μ_2. Since K has non-trivial trace, Theorem 9.33 shows that the adjoint map $\mu_2'\colon G_2 \to \mathrm{Hom}(H,K)$ is a quasi-split monomorphism. But μ_2 is simply the restriction to G_2 of the map $\mu'\colon G \to \mathrm{Hom}(H,K)$ adjoint to μ. Thus we have

$$G_2 \xrightarrow{\subseteq} G \xrightarrow{\mu'} \mathrm{Hom}(H,K)$$

and the composition of these two maps is a quasi-split monomorphism. By Proposition 3.* the inclusion $G_2 \hookrightarrow G$ is thus also a quasi-split monomorphism, i.e. G_2 is a quasi-summand of G. Since G is strongly indecomposable, we see that $G_2 = G$. But $G_2 \subseteq G_1 \subseteq G$, so $G_1 = G$.

2) Let $\psi\colon G \otimes K \to K$ be the quasi-projection. By Theorem 9.33 the adjoint map $\psi'\colon G \to \mathrm{Hom}(H,K)$ is monic. Since $\psi'(G_1) = 0$ it follows that $G_1 = 0$. ☑