3. QUASI-HOMOMORPHISMS

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Sometimes when one looks at finite rank torsion free modules one sees things that are almost true but not quite. A module may almost be a direct sum of two submodules, but a few elements don't fit. Or two modules may almost be isomorphic, but once again there are a few elements that get in the way.

Consider again Example 1.49. Using the concepts of Chapter 2 we can describe the construction as follows: We start with two rank-one modules A and B whose types $\mathbf{t}(A)$ and $\mathbf{t}(B)$ are incomparable (and also locally trivial, but that's not relevant at the moment.) We then take an element $w \neq 0 \in W$ such that there exist elements $a \in A$, $b \in B$ with $a \notin wA$, $b \notin wB$. We then construct a module H as the submodule of $QA \oplus QB$ generated by $A \oplus B$ and $(a+b)/w$. We can show that H is indecomposable. Indeed, the fact that $(a+b)/w \in H$ but $a/w \notin A$ and $b/w \notin B$ prevents H from being the direct sum of A and B. On the other hand, no other non-trivial direct sum decomposition can work because if $H = K \oplus L$ with $K, L \neq 0$ then K and L must be rank-one modules (since H has rank two) and by Proposition 2.* $\mathbf{t}(H) = \{\mathbf{t}(K), \mathbf{t}(L), \mathbf{t}(K) \wedge \mathbf{t}(L)\}$. But $\mathbf{t}(H) = \{\mathbf{t}(A), \mathbf{t}(B), \mathbf{t}(A) \wedge \mathbf{t}(B)\}$. Since $\mathbf{t}(A)$ and $\mathbf{t}(B)$ are incomparable, this can only work if $\mathbf{t}(K) = \mathbf{t}(A)$, $\mathbf{t}(L) = \mathbf{t}(B)$ or vice versa. But the only elements in H with type $\mathbf{t}(A)$ are those in A and the only elements with type $\mathbf{t}(B)$ are those in B, so this forces $K = A$ and $L = B$, or vice versa, the case which has already been ruled out.

In this example, the single element $(a+b)/w$ is the glue that prevents H from falling apart into the direct sum of A and B. Thus we have the sense that H is just barely indecomposable, and that the scantiness of the glue holding it together is in fact what makes it possible to play the trick that yields the non-unique direct decomposition in Example 1.49.

Contrast with this Example 1.45 as generalized in Example 2.28. Here we have three rank-one modules A, B, and C with mutually incomparable types and such that any pair have the same greatest lower bound, and we construct G as the submodule of Q generated by $A(1,0)$, $B(0,1)$ and $C(1,1)$, where $C(1,1)$, for instance, denotes $\{(c,c) \mid c \in C\}$. In contrast to the preceding example, the glue here is very strong, consisting of the infinite set of elements (c,c). (Or looked at another way, we have put an infinite number of different denominators under the element $(1,1)$.) In fact, it is really only a matter of point of view whether one thinks of G as being $A \oplus B$ held together with very strong glue consisting of all the elements (c,c) with $c \in C$, or whether G is based on $A \oplus C$ and the elements of $B(0,1)$ constitute the glue, or possibly G starts with $B \oplus C$ glued together with the elements of $A(1,0)$. In fact, the bonds holding G together are so strong that it only makes sense to think of it as a single unified whole. One should not expect to be able to use the module G to play dirty tricks with direct sum decompositions in the same way as was done in Example 1.49 using H.

Two modules with the same divisible hull which differ by only a finite number of elements, such as H and $A \oplus B$ do in Example 1.49, are called **quasi-equal**. (A more careful definition will be given below.) More generally, we say that G and H are **quasi-isomorphic** and we write $G \sim H$ if G is isomorphic to a module which is quasi-equal to H. The notion of quasi-isomorphism, invented by Bjarni Jónsson, is a very powerful tool for understanding direct sum decompositions and for gaining insight into finite rank torsion free modules in general. The following proposition shows that the definition of quasi-isomorphism just given is equivalent to the definition given in Chapter 2, where we said that two modules are quasi-isomorphic if each is isomorphic to a submodule of the other.

PROPOSITION 3.1. *Let G and H be finite rank torsion free modules. The following conditions are equivalent:*

1. *G is isomorphic to an essential submodule H' of H such that H/H' is finitely generated, i.e. $G \sim H$.*
2. *G is isomorphic to a submodule H' of H such that H/H' is not faithful.*
3. *G is isomorphic to a submodule H' of H such that H/H' has finite length.*
4. *There exist monomorphisms $G \to H$ and $H \to G$.*
5. *G is isomorphic to an essential submodule of H and H is a homomorphic image of $G \oplus F$ for some finitely generated W-module F.*

PROOF: (3) $\Rightarrow$ (1): Clear

(1) $\Rightarrow$ (2): (1) implies that H/H' is finitely generated and torsion, hence it is not faithful.

(2) $\Rightarrow$ (3): To say that H/H' is not faithful is to say that there exists $w \neq 0 \in W$ with $w(H/H') = 0$, i.e. $wH \subseteq H'$. Then H/H' is a homomorphic image of H/wH which has finite length by Proposition 1.22.

(2) $\Rightarrow$ (4): (2) implies a monomorphism $\theta\colon G \to H$ such that there exists $w \neq 0 \in W$ with $wH \subseteq \theta(G)$. Then we also get a monomorphism $H \to G$ by $h \mapsto \theta^{-1}(wh)$.

(4) $\Rightarrow$ (2): We may suppose that in fact $G \subseteq H$ and there is a monomorphism from H to G or, equivalently, there exists a monomorphism $\eta \in \operatorname{End} H$ with $\eta(H) \subseteq G$. Since η is monic it is clearly not a left zero divisor in $\operatorname{End} H$, so the left multiplication mapping $\varphi \mapsto \eta\varphi$ is a one-to-one mapping from $\operatorname{End} H$ to itself. Thus by Proposition 1.26 $\eta \operatorname{End} H$ is an essential submodule of $\operatorname{End} H$, so $W \cap \eta \operatorname{End} H \neq 0$. I.e. there exists $\varphi \in \operatorname{End} H$ and $w \neq 0 \in W$ such that $\eta\varphi = w$. Then $wH = \eta\varphi(H) \subseteq \eta(H) \subseteq G$ so that $w(H/G) = 0$ and H/G is not faithful.

(1) $\Rightarrow$ (5): Suppose that H' is an essential submodule of H such that there exists an isomorphism $\varphi_1\colon G \to H'$ and that H is generated by H' together with elements $h_1, \dots, h_n$. Then there is a map $\varphi_2\colon W^n \to H$ sending the i^{th} basis element of W^n onto h_i. Then φ_1 and φ_2 induce a homomorphism φ from $G \oplus W^n$ onto the submodule of H generated by H' and $h_1, \dots, h_n$, i.e. onto H.

(5) $\Rightarrow$ (1): **** ☑

THE CATEGORY OF TORSION FREE MODULES UNDER QUASI-HOMOMORPHISMS. Jónsson developed the concept of quasi-isomorphism in order to provide an environment in which finite rank torsion free abelian groups would have the Krull-Schmidt Property, i.e. direct sum decompositions would be unique up to (quasi-)isomorphism. Jónsson defined his concept within a framework of lattice theory, but later Jim Reid, among others, showed how to look at quasi-isomorphism from a ring-theoretic point of view, which is more comfortable for most abelian group theorists. The presentation here essentially follows Reid's.

In accord with our conventions, we denote by $\mathrm{QHom}(G,H)$ and $\mathrm{QEnd}\,G$ the divisible hulls of $\mathrm{Hom}(G,H)$ and $\mathrm{End}\,G$. We have $\dim \mathrm{QHom}(G,H) = \operatorname{rank} \mathrm{Hom}(G,H)$ and $\dim \mathrm{QEnd}\,G = \operatorname{rank} \mathrm{End}\,G < \infty$. Thus $\mathrm{QEnd}\,G$ is a finite dimensional Q-algebra. We then define the **category of finite rank torsion free W-modules under quasi-homomorphisms** as the category where the objects are finite rank torsion free modules and the morphisms from G to H are the elements of $\mathrm{QHom}(G,H)$. The maps in $\mathrm{QHom}(G,H)$ are called **quasi-homomorphisms** from G to H, and maps in $\mathrm{QEnd}\,G$ are called **quasi-endomorphisms** of G.

Recall that we usually make the identification $\mathrm{Hom}(G,H) \subseteq \mathrm{Hom}_Q(QG,QH)$. We then have also $\mathrm{QHom}(G,H) \subseteq \mathrm{Hom}_Q(QG,QH)$, and an instant's thought then shows that

A quasi-homomorphism φ from G to H is a linear transformation $\varphi\colon QG \to QH$ such that $w\varphi(G) \subseteq H$ for some $w \neq 0 \in W$.

PROPOSITION 3.2. *Two finite rank torsion free modules G and H are quasi-isomorphic if and only if they are isomorphic in the category of finite rank torsion free modules under quasi-homomorphisms.*

PROOF: ($\Rightarrow$): If $G \sim H$ then there exist monomorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Then $\varphi \in \mathrm{QHom}(G,H)$ and $\psi \in \mathrm{QHom}(H,G)$. It suffices to see that $\varphi\psi$ and $\psi\varphi$ have inverses in $\mathrm{QEnd}\,H$ and $\mathrm{QEnd}\,G$. But in fact the proof of (4) $\Rightarrow$ (2) in Proposition 3.1 shows that there exist $\eta \in \mathrm{QEnd}\,H$ and $\gamma \in \mathrm{QEnd}\,G$ such that $\eta\varphi\psi = w \neq 0 \in W$ and $\gamma\psi\varphi = w' \neq 0 \in W$, showing that $\varphi\psi$ is invertible in $\mathrm{QEnd}\,H$ and $\psi\varphi$ invertible in $QEndG$. (See also Proposition 3.4 below.)

($\Leftarrow$): If there exist $\varphi \in \mathrm{QHom}(G,H)$ and $\psi \in \mathrm{QHom}(H,G)$ with $\psi\varphi = 1_G$, $\varphi\psi = 1_H$, then φ and ψ must be monomorphisms, so $G \sim H$ by Proposition 3.1. ☑

We can restate Proposition 3.1 in another useful form. First, it will be useful to prove some properties that apply to any finite rank torsion free ring R. (Recall the convention that in this book except in Chapter 11 the word "ring" is used to mean W-algebra.) The following proposition is just a standard result in the theory of finite dimensional algebras once one notes that r is a left or right zero divisor in R if and only if the same is true in QR. In fact, if $sr = 0$ for some $s \in QR$, then there exists $w \neq 0 \in W$ with $ws \in R$ and $(ws)r = 0$.

PROPOSITION 3.3. *Let R be a finite rank torsion free ring and $r \in R$. The following conditions are equivalent:*

1. *r is not a left zero divisor in R.*
2. *r is not a right zero divisor in R.*
3. *r is left invertible in QR.*
4. *r is right invertible in QR.*

PROOF:

(3) $\Rightarrow$ (1): If $rs = 0$ and $tr = 1$ then $s = trs = 0$.

(4) $\Rightarrow$ (2): Analogous.

(1) $\Rightarrow$ (4): If r is not a left zero divisor in R then, as noted above, it is also not a left zero divisor in QR. Thus left multiplication by r is a monic linear transformation on the finite dimensional vector space QR, hence is an isomorphism, so $rQR = QR$. Thus $1 \in rQR$, i.e. there exists $s \in QR$ with $rs = 1$, so r is right invertible in QR.

(2) $\Rightarrow$ (3): Analogous. $\boxed{\checkmark}$

PROPOSITION 3.4. *Let $\varphi \in \operatorname{End} G$. The following conditions are equivalent:*

1. *φ is a monomorphism.*
2. *φ is not a left zero divisor in $\operatorname{End} G$.*
3. *φ is not a right zero divisor in $\operatorname{End} G$.*
4. *φ is left invertible in $\operatorname{QEnd} G$.*
5. *φ is right invertible in $\operatorname{QEnd} G$.*
6. *$G/\varphi(G)$ has finite length.*
7. *φ is a quasi-automorphism, i.e. there exists $\psi \in \operatorname{QEnd} G$ with $\varphi\psi = \psi\varphi = 1_G$.*

PROOF: (1) $\Rightarrow$ (2): If φ is monic then $\varphi\alpha = 0 \Rightarrow \alpha = 0$.

(4) & (5) $\iff$ (7): Clear.

(7) $\Rightarrow$ (1): Clear.

(5) $\Rightarrow$ (6): Let $\psi \in \operatorname{QEnd} G$ with $\varphi\psi = 1$ and let $0 \neq w \in W$ be such that $w\psi(G) \subseteq G$. Then $wG = w\varphi\psi(G) = \varphi(w\psi(G)) \subseteq \varphi(G)$ so that $G/\varphi(G)$ is a homomorphic image of G/wG. But by Proposition 1.22 G/wG has finite length, hence so does $G/\varphi(G)$.

(6) $\Rightarrow$ (1): If $\operatorname{length} G/\varphi(G) < \infty$ then $G/\varphi(G)$ is torsion so $\varphi(G)$ is an essential submodule of G and and φ is monic by Proposition 1.26.

The equivalence of (2) through (5) is a consequence of Proposition 3.3. $\boxed{\checkmark}$

PROPOSITION 3.5. *If R is a finite rank torsion free ring and $r \in R$, then r is left invertible in R if and only if r is right invertible in R.*

PROOF: Suppose that r is left invertible, say $sr = 1$ for $s \in R$. Then the proof of Proposition 3.4 shows that r has a two-sided inverse s' in QR. Then $s = s1 = srs' = 1s' = s'$ so that $rs = rs' = 1$ and s is in fact a two-sided inverse for r. $\boxed{\checkmark}$

QUASI-CONCEPTS. This book is dedicated to the point of view that for many purposes the category of modules under quasi-homomorphisms is *the* appropriate environment for studying finite rank torsion free modules. It is also true that there are many times when one jumps back and forth between this category and the more traditional category very quickly, even within a single sentence. The way to deftly accomplish this without confusion is by judicious use of the prefix "quasi."

1. We say that two modules G and H are **quasi-equal** if $QG = QH$ and the identity map on QG is a quasi-isomorphism from G to H and also from H to G. More prosaically, this means that there exist non-trivial elements w, $w' \in W$ such that $wG \subseteq H$ and $w'H \subseteq G$.
2. We say that a quasi-homomorphism $\varphi\colon G \to H$ is a **quasi-surjection** if H is quasi-equal to $\varphi(G)$.
3. By an (internal) **quasi-direct decomposition** of a module G we mean a pair of submodules H, $K \subseteq QG$ such that G is quasi-equal to $H \oplus K$. In this case, the associated projection $QG \to QH$ with kernel QK is in fact a quasi-homomorphism from G to H and is called the **quasi-projection** from G onto H.
4. By a **quasi-pure submodule** of G we mean a module $H \subseteq QG$ which is quasi-equal to a pure submodule of G.
5. We say that a pair of quasi-homomorphisms $H \xrightarrow{\alpha} G \xrightarrow{\beta} K$ is **quasi-exact** if $\operatorname{Ker}\beta$ is quasi-equal to $\alpha(H)$, and we say that a short exact sequence of quasi-homomorphisms $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ is **quasi-split** if there exists a quasi-homomorphism $\sigma\colon G \to H$ with $\sigma\alpha = 1_H$ or (equivalently) a quasi-homomorphism $\tau\colon K \to G$ with $\beta\tau = 1_K$. (See Proposition 3.16 below.)
6. We say that G is **strongly indecomposable** if there is no non-trivial quasi-direct decomposition of G. In other words, whenever G is quasi-equal to $H \oplus K$ then either H or K is trivial.

Note that isomorphic modules are quasi-isomorphic and strongly indecomposable modules are indecomposable.

EXAMPLE 3.6. The module H from Example 1.49, noted at the beginning of this chapter, where H is generated by $A \oplus B$ together with $(a+b)/w$, is merely indecomposable, not strongly indecomposable. On the other hand, the second example, G, generated by $A(1,0)$, $B(0,1)$ and $C(1,1)$, is strongly indecomposable.

One does not need a notion of "quasi-type" since invariance under quasi-isomorphisms is built into the very concept of type. In particular, one has the following result:

PROPOSITION 3.7. (1) *If $\varphi \in \operatorname{QHom}(G,H)$ then for every $g \in QG$, $\mathbf{t}_H(\varphi(g)) \geq \mathbf{t}_G(g)$.*
(2) *If H is a quasi-pure submodule of G then for every $h \in QH$, $\mathbf{t}_H(h) = \mathbf{t}_G(h)$.*

PROOF: (1) There exists $w \neq 0 \in W$ such that $w\varphi \in \operatorname{Hom}(G,H)$. Then $\mathbf{t}(\varphi(g)) = \mathbf{t}(w\varphi(g)) \geq \mathbf{t}(g)$ by Proposition 2.21.

(2) If H is quasi-pure in G then H is quasi-equal to a pure submodule H' of G. By Proposition 2.21 the type of an element $h \in QH$ with respect to H' is the same as its type with respect to G. Thus we may assume that $G = H'$, i.e. that H is quasi-equal to G. But this means that the identity map on QG belongs to both $\mathrm{QHom}(G,H)$ and $\mathrm{QHom}(H,G)$. Thus $\mathbf{t}_G(h) = \mathbf{t}_H(h)$ for all $h \in QH = QG$ by (1). $\boxed{\checkmark}$

The question arises as to whether it is permissible to use arrow notation to indicate quasi-homomorphisms. Your author is firmly committed to the belief that arrow notation is appropriate. Among other things, to disallow the use of arrows would be to disallow the use of commutative diagrams for quasi-homomorphisms, and this would definitely be a hardship. One the other hand, it has to be admitted that the indiscriminate use of arrows without indication as to what category the corresponding morphisms belong to can lead to a great deal of confusion. For this reason, in this book we have indicated quasi-homomorphisms by arrows only rarely, and only when there is a very clear indication that this is what we are doing.

As a further illustration of the idea that most of the statements one wants to make about finite rank torsion free modules can actually be made about quasi-equality classes, we have the following:

PROPOSITION 3.8. *If* $G \sim H$ *then*

1. $\operatorname{rank} G = \operatorname{rank} H$ *and for all prime ideals* p, $\text{p-rank}\, G = \text{p-rank}\, H$.
2. $\mathbf{d}(G) \approx \mathbf{d}(H)$.
3. $\mathbf{T}(G) = \mathbf{T}(H)$ *and* $\mathbf{CT}(G) = \mathbf{CT}(H)$.
4. $\operatorname{End} G$ *and* $\operatorname{End} H$ *are quasi-isomorphic.*
5. $\operatorname{QEnd} G$ *and* $\operatorname{QEnd} H$ *are isomorphic.*

PROOF: (1) From Proposition 3.1 it is clear that if $G \sim H$ then $QG \approx QH$, so $\operatorname{rank} G = \operatorname{rank} H$. Furthermore since G is isomorphic to an essential submodule of H by Proposition 1.24 $\text{p-rank}\, G \geq \text{p-rank}\, H$. Likewise $\text{p-rank}\, H \geq \text{p-rank}\, G$.

(2) By Proposition 3.1 we may assume that $G \subseteq H$ and for some $w \neq 0 \in W$, $wH \subseteq G$. Then $\mathbf{d}(wH) \subseteq \mathbf{d}(G) \subseteq \mathbf{d}(H)$. But it is easy to see that $\mathbf{d}(wH) = \mathbf{d}(H)$.

(3) Since we may assume without loss of generality that G and H are quasi-equal, this follows immediately from Proposition 3.7.

(4) With the same assumption as in (2), let $\varphi \in \operatorname{End} H$. Then $w\varphi(H) \subseteq G$ so a fortiori $w\varphi(G) \subseteq G$. Thus $w \operatorname{End} H \subseteq \operatorname{End} G$. On the other hand, if $\psi \in \operatorname{End} G$ then $w\psi(H) = \psi(wH) \subseteq \psi(G) \subseteq G \subseteq H$ so that $w\psi \in \operatorname{End} H$. Thus $w \operatorname{End} G \subseteq \operatorname{End} H$ so that $\operatorname{End} G$ and $\operatorname{End} H$ are quasi-equal.

(5) Immediate from (4) since quasi-isomorphic modules have isomorphic divisible hulls. $\boxed{\checkmark}$

PROPOSITION 3.9. *Suppose that* $G \sim H$.

1. *If* φ *is a monomorphism from* H *to* G, *then* φ *is a quasi-isomorphism.*
2. *If* $H \subseteq G$ *then* G *and* H *are quasi-equal.*

PROOF: (1) Let θ be a quasi-isomorphism from G to H. Then $\varphi\theta$ is a monomorphism in $\operatorname{QEnd} G$ and so by Proposition 3.4 is is a quasi-automorphism of G. Thus φ is quasi-surjective and hence a quasi-isomorphism.

(2) If $G \sim H$ and $H \subseteq G$ then apply (1) to 1_{QG} to see that G and H are quasi-equal. $\boxed{\checkmark}$

PROPOSITION 3.10. *If G is strongly indecomposable and I is a fractional ideal, then IG is strongly indecomposable.*

PROOF: If IG is quasi-equal to $H \oplus K$ then G is quasi-equal to $I^{-1}(H \oplus K) = I^{-1}H \oplus I^{-1}K$. $\boxed{\checkmark}$

PROPOSITION 3.11. *Let $H \subseteq G$. Then The following conditions are equivalent:*

1. *H is quasi-equal to G.*
2. *G/H is not faithful.*
3. *G/H has finite length.*

PROOF: (1) $\Leftrightarrow$ Since $H \subseteq G$, H is quasi-equal to G if and only if there exists $w \neq 0 \in W$ such that $wG \subseteq H$, i.e. $w(G/H) = 0$.

(2) $\Leftrightarrow$ (3) As seen in the proof of Proposition 3.1 (2) is equivalent to G/H having finite length. $\boxed{\checkmark}$

PROPOSITION 3.12. (1) *Let $\varphi\colon QG \to QH$. Then φ is a quasi-homomorphism from G to H if and only if $\varphi \in \operatorname{QHom}(G_p, H_p)$ for all primes p and $\varphi \in \operatorname{Hom}(G_p, H_p)$ for all but finitely many p.*

(2) *Let $QG = QH$. Then G and H are quasi-equal if and only if G_p and H_p are quasi-equal for all primes p and $G_p = H_p$ for all but finitely many p.*

PROOF: (1) ($\Rightarrow$): If $\varphi \in \operatorname{QHom}(G, H)$ then for some $w \neq 0 \in W$, $w\varphi(G) \subseteq H$. Then also $w\varphi(G_p) \subseteq H_p$ for all p, so $\varphi \in \operatorname{QHom}(G_p, H_p)$ for all p. Furthermore for all p with $w \notin p$, $wH_p = H_p$ and so $\varphi(G_p) \subseteq w^{-1}H_p = H_p$ so $\varphi \in \operatorname{Hom}(G_p, H_p)$. Since by Proposition 0.* there are only finitely many p with $w \in p$, it follows that $\varphi \in \operatorname{Hom}(G_p, H_p)$ for all but finitely many primes p.

($\Leftarrow$): For each p such that $\varphi \notin \operatorname{Hom}(G_p, H_p)$ there exists $w_p \neq 0 \in W$ with $w_p\varphi(G_p) \subseteq H_p$ because by assumption $\varphi \in \operatorname{QHom}(G_p, H_p)$. If there are only finitely many such p then we can multiply together the w_p to get a single $w \neq 0 \in W$ such that $w\varphi(G_p) \subseteq H_p$ for all the exceptional p. Clearly this also holds for the non-exceptional p. Thus $w\varphi(G) \subseteq H$ (see Proposition 0.*) so $\varphi \in \operatorname{QHom}(G, H)$.

(2) This follows immediately by applying (1) to the identity map on QG. $\boxed{\checkmark}$

The following triviality is used so pervasively in the remaining chapters that we have elevated it to the status of a lemma.

LEMMA 3.13. *Let R be a W-algebra and let M and N be R-modules which are torsion free as W-modules and such that M is finitely generated over R. Then*

1. $\mathrm{QHom}_R(M, N) = \mathrm{Hom}_R(QM, QN)$.
2. *If N is a finitely generated essential R-submodule of QM then M and N are quasi-equal (as W-modules.)*
3. *If R is a subring of S and S an essential subring of T such that T is finitely generated as an R-module, then S is quasi-equal to T.*

PROOF: We need to show that $\mathrm{Hom}_R(QM, QN) \subseteq Q\,\mathrm{Hom}_R(M, N)$, where the latter denotes the divisible hull of $\mathrm{Hom}_R(M, N)$ *as a W-module*. In fact, if $\varphi\colon QM \to QN$ is R-linear then there exists $w \neq 0 \in W$ such that $w\varphi(m) \in N$ for some finite set of generators for M over R. Then $w\varphi(M) \subseteq N$ so $\varphi \in \mathrm{QHom}_R(M, N)$.

In particular, if N is an R-submodule of QM with $QN = QM$, then this shows that $1_{QM} \in \mathrm{QHom}_R(M, N)$. If N is also finitely generated as an R-module, then we also see that 1_{QM} is a quasi-homomorphism from N to M. This means that M and N are quasi-equal. $\boxed{\checkmark}$

The analog of Proposition 1.8 is valid for quasi-pure modules. In particular, we need to note the following:

PROPOSITION 3.14. *If $H \lhd G$ and $H \subseteq K \subseteq G$ then K is a quasi-pure submodule of G if and only if K/H is a quasi-pure submodule of G/H.*

PROOF: ($\Rightarrow$): If K is quasi-pure in G then K is quasi-equal to a pure submodule K' of G. Then $K' = G \cap QK' = G \cap QK$ so that $H \subseteq K \subseteq K'$, and there exists $w \neq 0 \in W$ such that $wK' \subseteq K$. By Proposition 1.8 K'/H is pure in G/H. Since $w(K'/H) \subseteq K/H \subseteq K'/H$, it follows that K/H is quasi-equal to K'/H. Thus K/H is a quasi-pure submodule of G/H.

($\Leftarrow$): If K/H is quasi-pure in G/H, then K/H is quasi-equal to a pure submodule of G/H, which must have the form K'/H for some K' such that $H \subseteq K' \subseteq G$. Furthermore as in the preceding part of the proof $K/H \subseteq K'/H$, so that $K \subseteq K'$. By Proposition 1.8 $K' \lhd G$. Now there exists $w \neq 0 \in W$ such that $w(K'/H) \subseteq K/H$. Then $wK' \subseteq K \subseteq K'$, so K is quasi-equal to K'. Thus K is a quasi-pure submodule of G. $\boxed{\checkmark}$

Another result we'll need later is the following:

PROPOSITION 3.15. (1) *If $G = G_1 + G_2$ and $\varphi \in \mathrm{Hom}(G, QH)$ for any H, then φ is a quasi-homomorphism from G to H if and only if the restrictions of φ to G_1 and G_2 are quasi-homomorphisms from G_1 and G_2 to H.*

(2) *If in addition $QG_1 = QG_2 = QG$ then*

$$\mathrm{QHom}(G_1 + G_2, H) = \mathrm{QHom}(G_1, H) \cap \mathrm{QHom}(G_2, H).$$

PROOF: (1) If φ is a linear transformation from $QG = QG_1 + QG_2$ to QH and $\varphi \in \mathrm{QHom}(G, H)$ then for some $w \neq 0 \in W$, $w\varphi(G_1 + G_2) \subseteq H$ and so $w\varphi(G_1) \subseteq H$ and $w\varphi(G_2) \subseteq H$ so that $\varphi \in \mathrm{QHom}(G_1, H) \cap \mathrm{QHom}(G_2, H)$. On the other hand if there exist $w_1, w_2 \neq 0 \in W$ such that $w_1\varphi(G_1) \subseteq H$ and $w_2\varphi(G_2) \subseteq H$, then $w_1w_2\varphi(G_1 + G_2) = w_1\varphi(G_1) + w_2\varphi(G_2) \subseteq H$ and $\varphi \in \mathrm{QHom}(G_1 + G_2, H)$.

(2) This is simply a restatement of (1) since if $QG_1 = QG_2 = QG$ then by our conventions $\mathrm{QHom}(G, H)$, $\mathrm{QHom}(G_1, H)$ and $\mathrm{QHom}(G_2, H)$ are all identified as submodules of $\mathrm{Hom}(QG, QH)$. $\boxed{\checkmark}$

QUASI-DIRECT DECOMPOSITIONS. We will not bother to formalize such obvious facts as the fact that the composition of two quasi-homomorphisms is a quasi-homomorphism, or that if G is quasi-equal to G' and H to H' and $\varphi \in \mathrm{QHom}(G, H)$ then $\varphi \in \mathrm{QHom}(G', H')$. The point is that the "quasi" terminology works surprisingly well, and one can fearlessly take sentences which are known to be true for the usual category of modules under homomorphisms and change them into equally true sentences simply by inserting the word "quasi" throughout. For instance all the traditional theorems on direct sum decompositions have analogs for quasi-direct decompositions. The few proofs that follow should be enough to convince skeptics.

PROPOSITION 3.16. *Let H and K be submodules of G. The following conditions are equivalent:*

(1) *G is quasi-equal to $H \oplus K$.*
(2) *There exists a unique idempotent $\gamma' \in \mathrm{QEnd}\, G$ such that H is quasi-equal to $\gamma'(G)$. and K is quasi-equal to $\mathrm{Ker}\, \gamma'$.*
(3) *There exists $\gamma \in \mathrm{End}\, G$ and $w \neq 0 \in W$ such that $\gamma^2 = w\gamma$ and H is quasi-equal to $\gamma(G)$ and K is quasi-equal to $\mathrm{Ker}\, \gamma$.*

PROOF: (1) $\Rightarrow$ (2): If G is quasi-equal to $H \oplus K$, then $QG = QH \oplus QK$. Let γ be the unique idempotent linear transformation on QG with image QH and kernel QK. Let $w, w' \neq 0 \in W$ be such that $wG \subseteq H \oplus K$ and $w'(H \oplus K) \subseteq G$. Then $w\gamma(G) = \gamma(wG) \subseteq \gamma(H \oplus K) \subseteq H$. Then $ww'\gamma(G) \subseteq w'H \subseteq G$, so $\gamma \in \mathrm{QEnd}\, G$. Since also $w'H = w'\gamma(H \oplus K) \subseteq \gamma(G)$, $\gamma(G)$ is quasi-equal to H. Now as an endomorphism of QG, γ has kernel QK. But since we are thinking of γ as a mapping with domain G, we have $\mathrm{Ker}\, \gamma = G \cap QK \subseteq G$. Then if w and w' are as above, $w'K \subseteq G \cap QK = \mathrm{Ker}\, \gamma$ and $w(\mathrm{Ker}\, \gamma) \subseteq QK \cap (H \oplus K) = K$. Thus $\mathrm{Ker}\, \gamma$ is quasi-equal to K.

(2) $\Rightarrow$ (3): If $\gamma' \in \mathrm{QEnd}\, G$ is the idempotent above then let $\gamma = w\gamma'$, where $w \neq 0 \in W$ is chosen so that $\gamma \in \mathrm{End}\, G$. Then $\gamma^2 = w^2{\gamma'}^2 = w\gamma$ and H is quasi-equal to $\gamma(G)$, K is quasi-equal to $\mathrm{Ker}\, \gamma$.

(3) $\Rightarrow$ (1): It will suffice to prove that following more general Lemma.

LEMMA 3.17. *Let $\gamma \in \mathrm{End}\, G$ and let $w \neq 0 \in W$ be such that $\gamma^2 = w\gamma$. Then $wG \subseteq \gamma(G) \oplus \mathrm{Ker}\, \gamma \subseteq G$.*

PROOF: $\mathrm{Ker}\, \gamma \cap \gamma(G) = 0$ since if $x = \gamma(g)$ and $\gamma(x) = 0$ then $wx = w\gamma(g) = \gamma^2(g) = \gamma(x) = 0$ so $x = 0$. Now for any $g \in G$, $wg = \gamma(g) + (wg - \gamma(g))$, where $\gamma(g) \in \gamma(G)$ and $wg - \gamma(g) \in \operatorname{Ker} \gamma$ since $\gamma(wg - \gamma(g)) = w\gamma(g) - \gamma^2(g) = 0$. Thus $wG \subseteq \gamma(G) \oplus \operatorname{Ker} \gamma \subseteq G$. $\boxed{\checkmark}$

PROPOSITION 3.18. *Let* $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ *be a short quasi-exact sequence. Then* $\alpha(H)$ *is a quasi-summand of* G *if and only if this sequence is quasi-split. More precisely, if we assume that* $\alpha \in \operatorname{Hom}(H, G)$ *and* $\beta \in \operatorname{Hom}(G, K)$ *and* $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ *is actually exact, then for any* $w \neq 0 \in W$ *the following conditions are equivalent:*

1. *There exists* $\sigma \in \operatorname{Hom}(G, H)$ *such that* $\sigma\alpha = w$.
2. *There exists* $\tau \in \operatorname{Hom}(K, G)$ *such that* $\beta\tau = w$.
3. *There exists* $L \subseteq G$ *with* $\alpha(H) \cap L = 0$ *and* $wG \subseteq \alpha(H) \oplus L$.
4. *There exists* $L \subseteq G$ *such that* $\alpha(H) \cap L = 0$ *and* $\beta(L)$ *is quasi-equal to* K.
5. $\alpha(H)$ *is a direct summand of* $\alpha(H) + wG$.

PROOF: First note that if $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ is quasi-exact then there exists H' quasi-equal to H and K' quasi-equal to K such that $\alpha(H') = \operatorname{Ker} \beta \subseteq G$ and $\beta(G) = K'$, so that there is no loss of generality in assuming that α and β are honest homomorphisms and that the sequence is actually exact. Note also that $\alpha(H)$ is a quasi-summand of G if and only if (3) is true for some $w \neq 0 \in W$, and that the sequence is quasi-split if and only if (1) and (2) are true for some $w \neq 0 \in W$. Thus it suffices to prove the equivalence of (1) through (4).

(1) ⇒ (3): If there exists $\sigma \colon G \to H$ such that $\sigma\alpha = w$ then let σ' be defined on $\alpha(H) + wG$ by $\sigma'(x) = \sigma(x/w) = \sigma(x)/w$. Then $\sigma'(\alpha(h)) = \sigma(\alpha(h))/w = h$ for $h \in H$, and in particular $\sigma'(\alpha(H)) \subseteq H$. Also $\sigma'(wG) = \sigma(G) \subseteq H$. Thus $\sigma'(\alpha(H) + wG) \subseteq H$ so that $\sigma' \in \operatorname{Hom}(\alpha(H) + wG,\, H)$ and $\sigma'\alpha = 1_H$, so σ' splits α. Thus $\alpha(H)$ is a direct summand of $\alpha(H) + wG$.

(2) ⇒ (3): Suppose that there exists $\tau \colon K \to G$ with $\beta\tau = w$. Now define $\beta' \in \operatorname{Hom}(\alpha(H) + wG,\, K)$ by $\beta'(x) = \beta(x)/w$. Then $\operatorname{Ker} \beta' = \operatorname{Ker} \beta = \alpha(H)$ and $\beta'(\alpha(H) + wG) = \beta'(wG) \subseteq K$. Since $\beta'\tau = 1_K$, β' is split and so $\alpha(H)$ is a direct summand of $\alpha(H) + wG$.

(3) ⇔ (4): ****

(3) ⇒ (5): Given $L \subseteq G$ with $wG \subseteq \alpha(H) \oplus L \subseteq G$, then $\alpha(H) \subseteq \alpha(H) + wG \subseteq \alpha(H) \oplus L$. Since $\alpha(H)$ is a direct summand of $\alpha(H) \oplus L$ it follows that $\alpha(H)$ is a direct summand of $\alpha(H) + wG$.

(5) ⇒ (1): If $\alpha(H)$ is a direct summand of $\alpha(H) + wG$ then there is is a splitting map $\sigma' \in \operatorname{Hom}(\alpha(H) + wG,\, H)$ with $\sigma'\alpha = 1_H$. Let $\sigma = w\sigma'$. Then $\sigma(G) = w\sigma'(G) = \sigma'(wG) \subseteq H$ so that $\sigma \in \operatorname{Hom}(G, H)$ and $\sigma\alpha = w$.

(4) ⇒ (2): If $\alpha(H)$ is a summand of $\alpha(H) + wG$ then there exists $\tau' \colon \beta(\alpha(H) + wG) = wK \to \alpha(H) + wG$ such that $\beta\tau' = 1_{wK}$. Let $\tau = w\tau'$. Then $\tau(K) = w\tau'(K) = \tau'(wK) \subseteq G$ so that $\tau \in \operatorname{Hom}(K, G)$ and $\beta\tau = w$. $\boxed{\checkmark}$

EXAMPLE 3.19. Consider again Example 1.49, as reformulated at the beginning of this chapter. The module G is generated by $A \oplus B$ together with $(a + b)/w$ for a certain $w \in W$, where $a \notin wA$, $b \notin wB$. The map $\gamma \colon Q \oplus Q \to Q$ given by $(x, y) \mapsto wy$ maps G

onto the submodule B' of B generated by wB and b. Since $wG \subseteq A \oplus B$ it follows that the short exact sequence

$$0 \to A \to G \to B' \to 0$$

is quasi-split. We can also notice that $A + wG$ is generated by A and wB together with $a + b$. Thus $b \in A + wG$. But b generates B' modulo wB so that $A + wG = A \oplus B'$, so that A is a summand of $A + wG$, confirming the proposition. We have $\sigma \in \operatorname{Hom}(G, A)$ given by $\sigma(a, b) = wa$ and $\tau \in \operatorname{Hom}(B', G)$ given by $\tau(b) = b$ and if we let $\alpha\colon A \to G$ be the inclusion map then $\sigma\alpha = w$ and $\gamma\tau = w$.

It turns out that any module H with rank two which is indecomposable but not strongly indecomposable looks pretty much like the H in Example 3.19.

PROPOSITION 3.20. *Let H be an indecomposable module with* $\operatorname{rank} H = 2$ *such that H is not strongly indecomposable. Then there exist pure rank-one submodules A, B of H and elements $a \in A$, $b \in B$ and $q \in Q$ such that H is generated by $A \oplus B$ together with $q(a + b)$. Furthermore $H/(A \oplus B) \approx A/IA \approx B/IB \approx W/I$, where I is the ideal in W such that I^{-1} is the fractional ideal generated by 1 and q. If W is a principal ideal domain then we may choose $q = 1/w$ for some $w \in W$.*

PROOF: Since H is not strongly indecomposable there exist non-trivial submodules A, $B \subseteq H$ such that H is quasi-equal to $A \oplus B$. Since $\operatorname{rank} H = 2$, A and B must have rank one. We may assume that A and B are pure in H since we can replace them by their purifications.

Let α and β be the quasi-projections of H onto A and B, and let $A' = \alpha(H)$, $B' = \beta(H)$. Then $A \subseteq A'$, $B \subseteq B'$ and A is quasi-equal to B', B is quasi-equal to B'. By Proposition 2.2 $A'/A \approx W/I$ for some ideal I and $A = IA'$. Now α induces a surjection $\bar\alpha\colon H \to A'/A$ and we claim that $\operatorname{Ker}\bar\alpha = A \oplus B$. In fact, let $h = a + b \in H$ with $a \in QA$, $b \in QB$. Then the assumption that $A \lhd H$ yields $h \in \operatorname{Ker}\bar\alpha \iff \alpha(h) = a \in A \iff a \in H \iff b = h - a \in H \iff h \in A \oplus B$. Thus $H/(A \oplus B) \approx A'/A \approx W/I$ and likewise $H/(A \oplus B) \approx B'/B$, so that $B'/B \approx W/I$. It follows that $B = IB'$. Now $1 \in I^{-1}$ (since $I \subseteq W$) so by Proposition 0.* I^{-1} is generated by 1 together with some $q \in I$. If I is principal, generated by $w \in W$, then we may choose $q = 1/w$. Then $A \oplus B \subseteq H \subseteq A' \oplus B' = I^{-1}(A \oplus B) = (A \oplus B) + q(A \oplus B)$. Since $H/(A \oplus B)$ is cyclic, H is generated by $A \oplus B$ together with some single element $h \in q(A \oplus B)$, so $h = q(a + b)$ for some $a \in A$, $b \in B$. $\boxed{\checkmark}$

Another familiar theorem about direct sum decompositions which carries over to quasi-direct compositions is the following:

PROPOSITION 3.21. *Let $H \subseteq K \subseteq G$.*

1. *If H is a quasi-summand of G then H is a quasi-summand of K. In fact, if G is quasi-equal to $H \oplus L$ then K is quasi-equal to $H \oplus (L \cap K)$.*
2. *If $H \lhd G$ and K is a quasi-summand of G then K/H is a quasi-summand of G/H.*
3. *If H is a quasi-summand of G then H is quasi-pure in G.*

PROOF: (1) If H is a quasi-summand of G then by Proposition 3.16 H is quasi-equal to $\gamma(G)$ for some idempotent $\gamma \in \operatorname{QEnd} G$. Since $\gamma(QG) = QH \subseteq QK$, γ restricts to a linear endomorphism γ' defined on QK. Now because γ is idempotent, $H = \gamma(H) \subseteq \gamma(K) \subseteq \gamma(G)$. But H is quasi-equal to $\gamma(G)$, so it follows that H is equal to $\gamma(K) = \gamma'(K)$. Thus $\gamma' \in \operatorname{QEnd} K$. Thus γ' is an idempotent quasi-endomorphism of K. By Proposition 3.16 K is quasi-equal to $H \oplus \operatorname{Ker} \gamma'$. But $\operatorname{Ker} \gamma' = K \cap \operatorname{Ker} \gamma$, which is quasi-equal to $K \cap L$ since $\operatorname{Ker} \gamma$ is quasi-equal L.

(2) A quasi-splitting map $G/K \to G$ induces a quasi-splitting map $(G/H)/(K/H) = G/K \to G/H$.

(3) If G is quasi-equal to $H \oplus K$ and w, $w' \neq 0 \in W$ are such that $wG \subseteq H \oplus K$ and $w'(H \oplus K) \subseteq G$, then $w(G \cap QH) \subseteq H$ and $w'H \subseteq G \cap QH$, so that H is quasi-equal to the pure submodule $G \cap QH$ of G. $\boxed{\checkmark}$

JÓNSSON's THEOREM. The brilliant idea of looking at the category of finite rank torsion free W-modules under quasi-isomorphisms is that we are able to work in a category where the homomorphism groups are all finite dimensional vector spaces. Since this is what really makes most of the classical theorems about modules over finite dimensional algebras work, we can expect most of these theorems to carry over into our category. In particular, we get the Jónsson's analog of the Krull-Schmidt Theorem:

THEOREM 3.22. *The representation of a module as a quasi-direct sum of strongly indecomposable modules is unique up to quasi-isomorphism.*

PROOF: We can simply follow the standard Azamaya proof, once we have established the following:

PROPOSITION 3.23. *If G is strongly indecomposable, then $\operatorname{QEnd} G$ is a local ring.*

PROOF: The proof is still classical. Namely, a finite dimensional algebra without proper idempotents is a local ring. (In fact if $QN = \operatorname{nil\,rad} \operatorname{QEnd} G$ then $(\operatorname{QEnd} G)/QN$ can have no proper idempotents, since idempotents modulo QN lift to idempotents in $\operatorname{QEnd} G$ by Proposition 1.32. Thus $(\operatorname{QEnd} G)/QN$ is a semi-simple ring without idempotents, hence is a skew field, in particular has no proper left ideals. It follows that QN is the unique maximal left ideal in $\operatorname{QEnd} G$.)

It thus suffices to see that if G is strongly indecomposable then $\operatorname{QEnd} G$ has no proper idempotents, which is equivalent to saying that if $\operatorname{QEnd} G$ has a proper idempotent then G has a non-trivial quasi-decomposition. But we have already proved this as Proposition 3.16. $\boxed{\checkmark}$

For those who non-commutative ring theory is weak, an alternative proof of Proposition 3.23 occurs further below (Corollary 3.26).

PROOF OF JÓNSSON'S THEOREM. We will actually prove a stronger result:

THEOREM 3.24. *Let G be quasi-equal to $G_1 \oplus \cdots \oplus G_s$ and also to $H_1 \oplus \cdots \oplus H_t$, where the G_i and H_j are strongly indecomposable. Then $s = t$ and the H_j can be renumbered such that for each i, the summands G_i and H_i are interchangeable. In other words if for each $i = 1, \ldots, s$ we let X_i denote either G_i or H_i, then G is quasi-equal to $X_1 \oplus \cdots \oplus X_s$.*

PROOF: We may assume WLOG that $G = G_1 \oplus \ldots G_s$. Let γ_1 be the projection of G onto G_1 with kernel equal to $G_2 \oplus \cdots \oplus G_s$ and for $j = 1, \ldots, t$ let η_j be the quasi-idempotent in $\operatorname{QEnd} G$ with $\eta_j(G)$ quasi-equal to H_j. Let $\varphi_j \in \operatorname{QEnd} G_1$ be the restriction of $\gamma_1 \eta_j$ to G_1. Then $\varphi_1 + \cdots + \varphi_t = 1 \in \operatorname{QEnd} G_1$ since $\sum \eta_j = 1 \in \operatorname{QEnd} G$. Since $\operatorname{QEnd} G_1$ is a local ring, at least one of the φ_i, say φ_1, must be invertible (otherwise all $\varphi_i \in \operatorname{nil rad} \operatorname{QEnd} G$ which implies the contradiction $1 \in \operatorname{nil rad} \operatorname{QEnd} G$). By renumbering the H_j we may suppose that φ_1 is invertible, i.e. $\gamma_1 \eta_1$ restricts to an quasi-automorphism of G_1. By Proposition 3.18 this means that η_1 restricts to a split quasi-monomorphism from G_1 onto a quasi-summand of H_1. But by assumption H_1 is strongly indecomposable, so that η_1 must thus restrict to a quasi-isomorphism from G_1 to H_1 and γ_1 to a quasi-isomorphism from H_1 to G_1. Thus G_1 is quasi-equal to $\gamma_1(H_1)$, so G is quasi-equal to $\gamma_1(H_1) \oplus G_2 \oplus \cdots \oplus G_s$.

We now claim that $\gamma_1(H_1) \oplus G_2 \cdots \oplus G_s$ is quasi-equal to $H_1 \oplus G_2 \oplus \cdots \oplus G_s$. First note that since γ_1 restricted to H_1 is monic, then $H_1 \cap (G_2 \oplus \cdots \oplus G_s) = H_1 \cap \operatorname{Ker} \gamma_1 = 0$, so that the sum $H_1 + G_2 \oplus \cdots \oplus G_s$ is in fact direct. Now let $w \neq 0 \in W$ be such that $wH_1 \subseteq G = G_1 \oplus \cdots \oplus G_s$. If $g_1 \in \gamma_1(H_1)$ then there exists $h_1 \in H_1$ with $g_1 = \gamma_1(h_1)$, and what this means is that $wg_1 = wh_1 + x$ for some $x \in G_2 \oplus \cdots \oplus G_s$. Thus $w\gamma_1(H_1) \subseteq wH_1 \oplus G_2 \oplus \cdots \oplus G_s$ and so $w\gamma_1(H_1) \oplus G_2 \oplus \cdots \oplus G_s \subseteq wH_1 \oplus G_2 \oplus \cdots \oplus G_s \subseteq G$. Since we have seen that G is quasi-equal to $\gamma_1(H_1) \oplus G_2 \oplus \cdots \oplus G_s$, this shows that G is quasi-equal to $H_1 \oplus G_2 \oplus \cdots \oplus G_s$, so that H_1 and G_1 are interchangeable up to quasi-equality.

We now have $G_1 \oplus G_2 \oplus \cdots \oplus G_s$ quasi-equal to $G_1 \oplus H_2 \oplus \cdots \oplus H_t$. Applying the same argument again to G_2 shows that G_2 is interchangeable with either G_1 or H_i for some $i = 2, \ldots, t$. But clearly G_2 and G_1 are not interchangeable. Thus we may renumber the H_i so that G_2 and H_2 are interchangeable. Continuing in this fashion, we obtain the theorem. $\boxed{\checkmark}$

MORE STOLEN THEOREMS. There are other theorems we can steal from the classical theory of finite dimensional algebras. The following result is known in the classical theory as Fitting's Lemma.

THEOREM 3.25. *Let $\gamma \in \operatorname{End} G$. Then G is quasi-equal to $H \oplus K$, where H and K are invariant under γ and the restriction of γ to H is a quasi-automorphism and $\gamma^n(K) = 0$ for some $n \geq 1$.*

PROOF: The ascending chain of pure submodules $\operatorname{Ker} \gamma \subseteq \operatorname{Ker} \gamma^2 \subseteq \ldots$ must become stationary at some $n \leq \operatorname{rank} G$. Let $K = \operatorname{Ker} \gamma^n$ and let $H = \gamma^n(G)$. Then clearly H and K are invariant under γ and by definition $\gamma^n(K) = 0$. Furthermore, $H \cap \operatorname{Ker} \gamma = 0$: in fact, if $h \in H \cap \operatorname{Ker} \gamma$ then $(\exists g)\, h = \gamma^n(g)$ and $\gamma(h) = \gamma^{n+1}(g) = 0$ so $g \in \operatorname{Ker} \gamma^{n+1} = \operatorname{Ker} \gamma^n$ so $h = \gamma^n(g) = 0$. Thus the restriction of γ to H is monic, hence is a quasi-automorphism, as required. Now let $\theta \in \operatorname{QEnd} H$ be an inverse to the restriction of γ^n. Then $\theta\gamma^n \in \mathrm{QHom}(G, H)$ and $\theta\gamma^n$ restricts to the identity on H, hence $\theta\gamma^n$ is a quasi-projection and G is quasi-equal to $H \oplus \mathrm{Ker}(\theta\gamma^n)$. But $\mathrm{Ker}\,\theta\gamma^n = \mathrm{Ker}\,\gamma^n = K$. $\boxed{\checkmark}$

From this we get an extremely useful corollary. (In fact your author ventures that he has probably got more mileage out of the following result than from any other theorem in the subject.)

COROLLARY 3.26. *G is strongly indecomposable if and only if every (quasi-) endomorphism of G is either monic or belongs to* $\mathrm{nil\,rad\,QEnd}\,G$. *In particular, this shows (again) that* $\mathrm{QEnd}\,G$ *is a local ring.*

PROOF: ($\Leftarrow$): If every endomorphism of G is either monic or belongs to the nil radical of $\mathrm{QEnd}\,G$ then $\mathrm{QEnd}\,G$ can have no proper idempotents, because if γ is a proper idempotent then $(\exists w \neq 0 \in W)\, w\gamma \in \mathrm{End}\,G$ and $w\gamma$ is neither monic nor nilpotent.

($\Rightarrow$): If $\gamma \in \mathrm{End}\,G$, apply Proposition 3.25. Since G has no proper quasi-direct decomposition, either γ is a quasi-automorphism or $G = \mathrm{Ker}\,\gamma^n$ so that γ is nilpotent. Furthermore if γ is not a quasi-automorphism then by Propositioin 3.* it is not monic, and therefore for every $\beta \in \mathrm{QEnd}\,G$, $\beta\gamma$ is not a quasi-automorphism, and therefore must also be nilpotent. Thus the left ideal generated by γ contains only nilpotent elements, hence $\gamma \in \mathrm{nil\,rad\,QEnd}\,G$.

Note that this shows that $\mathrm{nil\,rad\,QEnd}\,G$ is a maximal left ideal (and necessarily the unique one), since any element not in the nilradical is invertible in $\mathrm{QEnd}\,G$. Thus $\mathrm{QEnd}\,G$ is a local ring. $\boxed{\checkmark}$

COROLLARY 3.27. *If G is strongly indecomposable and has no nilpotent endomorphisms, then* $\mathrm{rank}(\mathrm{End}\,G)$ *divides* $\mathrm{rank}\,G$.

PROOF: By the Corollary 3.26 every non-trivial endomorphism of G is monic, hence invertible in $\mathrm{QEnd}\,G$, and it follows that $\mathrm{QEnd}\,G$ is a skew field. Now QG is a vector space over this skew field, hence $\mathrm{rank}\,G = \dim QG$ must be a multiple of $\dim \mathrm{QEnd}\,G = \mathrm{rank}\,\mathrm{End}\,G$. $\boxed{\checkmark}$

COROLLARY 3.28. *Let G be strongly indecomposable and* $N = \mathrm{nil\,rad\,End}\,G$. *Then* $N = \{\varphi \in \mathrm{End}\,G \mid \varphi(G) \subseteq NG\}$.

PROOF: Clearly $\varphi \in N \Rightarrow \varphi(G) \subseteq NG$. Conversely, by Proposition 1.33 $\mathrm{rank}\,NG < \mathrm{rank}\,G$. Thus if $\varphi(G) \subseteq NG$ then φ is not monic, and for all $\gamma \in \mathrm{End}\,G$, $\gamma\varphi$ is not monic. Thus by Corollary 3.26 $\gamma\varphi$ is nilpotent. Thus the left ideal in $\mathrm{End}\,G$ generated by φ is nil and it follows by Proposition 1.30 that $\varphi \in N$. $\boxed{\checkmark}$

The following corollary to Fitting's Lemma would be trivial if the hypothesis were that G be p-reduced, but is surprising since we require only the much weaker assumption that G has no p-divisible quasi-summand. It is also surprising that the hypothesis only imposes a condition relevant to a single prime p.

PROPOSITION 3.29. *Let p be a prime ideal and G a module with no p-divisible quasi-summand. Let $\varphi \in \operatorname{End} G$. If φ preserves p-heights then φ is monic.*

PROOF: Assume that φ preserves p-heights and apply Proposition 3.25 to get a quasi-decomposition of G as $H \oplus K$ where φ restricts to a quasi-automorphism of H and $K = \operatorname{Ker} \varphi^n$ for some n. Since φ preserves p-heights, if $g \in \operatorname{Ker} \varphi$ then $\mathrm{ht}_p(g) = \mathrm{ht}_p(0) = \infty$. By induction we see that $\mathrm{ht}_p(g) = \infty$ for all $g \in \operatorname{Ker} \varphi^n = K$, so K is p-divisible. By assumption, then, $K = 0$, so φ is a quasi-automorphism. $\boxed{\checkmark}$

MURLEY MODULES. We noted in Chapter Two that rank-one modules are the atoms of the theory of finite rank torsion free modules. If we restrict attention to homomorphisms of *reduced* modules, though, we find that indecomposable modules G with the property that p-rank $G \leq 1$ for every prime ideal p have many of the properties of rank-one modules.

DEFINITION 3.30. A **Murley module** is a reduced torsion free module G such that p-rank $G \leq 1$ for all prime ideals p. A **cohesive module** is a reduced finite rank torsion free module G such that every proper homomorphic image of G is divisible. A **Murley ring** is a torsion free W-algebra R such that R is a Murley module and a **cohesive ring** is a torsion free ring which is a cohesive module.

These definitions differ from the ones in [Arnold], [Murley], and [Dubois] by the added restriction that G be reduced. This avoids certain nuisance special cases. **We will not assume that Murley modules have finite rank, but we will require that a Murley module cannot be written as an infinite direct sum of non-trivial summands.**

PROPOSITION 3.31. *G is a cohesive module if and only if G is a Murley module and for all primes p, G is either p-divisible or p-reduced.*

PROOF: ($\Leftarrow$): If H is a pure submodule of G and p is a prime then by assumption either G is p-divisible, so p-rank $H =$ p-rank $G = 0$, or G is p-reduced and p-rank $G = 1$, so H is also p-reduced and $1 \leq$ p-rank $H \leq$ p-rank $G = 1$. In either case p-rank $G/H = 0$ so G/H is p-divisible. Since this is true for all p, G/H is divisible.

($\Rightarrow$): If on the one hand p-rank $G > 1$ for some p then G must contain a pure rank-one module A which is not p-divisible (otherwise G would be p-divisible, contradicting the assumption that p-rank $G > 1$). Then p-rank $A = 1 <$ p-rank G so that p-rank $G/A \neq 0$ and G/A is a proper homomorphic image of G which is not divisible. And if there exists p with $0 \neq p^{-\infty} G \subsetneq G$ then $G/p^{\infty} G$ is not divisible since it is not p-divisible. $\boxed{\checkmark}$

PROPOSITION 3.32. *G is a Murley module if and only if G is isomorphic to a pure submodule of $\prod_p \hat{W}_p$.*

PROOF: ($\Leftarrow$): If $G \quad \lhd \quad \prod \hat{W}_p$ then G is reduced and for all p, p-rank $G \leq$ p-rank $\prod_p \hat{W}_p = 1$.

($\Rightarrow$): Suppose that G is a Murley module. By Proposition 1.*, for each prime p the completion $\hat{G}_p$ is a free $\hat{W}_p$-module with the same p-rank as $\text{p-rank}\, G$, namely 1 or 0. Thus $\hat{G}_p = 0$ or $\hat{G}_p \approx \hat{W}_p$ for all p. Now the canonical maps $\theta_p \colon G \to \hat{G}_p$ induce a map $\theta \colon G \to \prod \hat{G}_p$. Furthermore $\operatorname{Ker} \theta = \bigcap \operatorname{Ker} \theta_p = \bigcap p^\infty G = \mathbf{d}(G) = 0$, so that θ is monic. Finally, for each prime p, $\theta_p(G_p) \triangleleft \hat{G}_p$ and it follows that $\theta(G)$ is p-pure in $\prod_p \hat{G}_p$. Since this is true for every p, $\theta(G)$ is a pure submodule of $\prod_p \hat{G}_p \approx \prod_p \hat{W}_p$. $\boxed{\checkmark}$

PROPOSITION 3.33. *G is a Murley module if and only if every finite length homomorphic image of G is cyclic or trivial.*

PROOF: ($\Leftarrow$): If every homomorphic image of G is cyclic, then G/pG is a cyclic W/p-module, hence is simple or trivial. Thus $\text{p-rank}\, G \leq 1$.

($\Rightarrow$): Let G be a Murley module and suppose by way of contradiction that there exists a non-trivial homomorphic image G/H which is not cyclic. Then by Proposition 0.* some p-primary component of G/H is not cyclic. This p-primary component is a homomorphic image of G/H and thus a homomorphic image of G. Therefore WLOG we may suppose that G/H is p-primary and not cyclic. But then there is a homomorphic image G/H' of G/H such that $p(G/H') = 0$ and G/H' is not cyclic. Then $pG \subseteq H'$, so G/pG is not cyclic. It follows that $\text{p-rank}\, G = \text{length}\, G/pG > 1$, a contradiction. $\boxed{\checkmark}$

Recall that the **reduced tensor product** of G and H is defined to be $G \star H = (G \otimes H)/\mathbf{d}(G \otimes H)$.

PROPOSITION 3.34. (1) *A pure submodule of a Murley module is a Murley module.*

(2) *If G and H are Murley modules then so are $G \star H$ and $\operatorname{Hom}(G, H)$.*

(3) *If G and H are cohesive then so is $\operatorname{Hom}(G, H)$.*

PROOF: (1) If $H \triangleleft G$ and G is a Murley module, then $\mathbf{d}(H) \subseteq \mathbf{d}(G) = 0$ so H is reduced. Furthermore for every prime p, $\text{p-rank}\, H \leq \text{p-rank}\, G \leq 1$. Thus H is a Murley module.

(2) If H is reduced then by Proposition 1.34 so is $\operatorname{Hom}(G, H)$. Furthermore by Proposition 1.36 $\text{p-rank} \operatorname{Hom}(G, H) \leq (\text{p-rank}\, G)(\text{p-rank}\, H) \leq 1$. Likewise $G \star H$ has p-rank at most one, and by definition $G \star H$ is reduced.

(3) By Proposition 1.34 if H is p-reduced or p-divisible then so is $\operatorname{Hom}(G, H)$. Since $\operatorname{Hom}(G, H)$ is Murley module by (1), the result follows from Proposition 3.27. $\boxed{\checkmark}$

Proposition 3.34 is stated in terms of $G \star H$ rather than $G \otimes H$ because $G \otimes H$ need not be reduced even when G and H are, as was seen in Example 2.36.

In general, the determination of $\mathbf{d}(G \otimes H)$ is a non-trivial problem. We will address this issue in Chapter 8.

The following Proposition shows a very important way that Murley modules are like rank-one modules.

PROPOSITION 3.35. *G is a Murley module if and only if whenever H is quasi-equal to G then $H = IG$ for some fractional ideal I. If $H \subseteq G$ then we may choose $I \subseteq W$.*

PROOF: ($\Leftarrow$): Suppose that G has the indicated property and let p be a prime. We must show that p-rank $G \leq 1$, i.e. that G/pG is a simple module. Now a submodule of G/pG has the form H/pG where $pG \subseteq H \subseteq G$. By hypothesis $H = IG$ for some fractional ideal I. Now recall from Proposition 2.11 that if I_1 and I_2 are submodules of Q then $(I_1 \cap I_2)G = I_1G \cap I_2G$ and $(I_1 + I_2)G = I_1G + I_2G$. From this we see that $(I \cap W)G = IG \cap G = IG = H$, so we lose no generality in supposing $I \subseteq W$. Furthermore we have $(I + p)G = IG + pG = H + pG = H$ so we may as well suppose $p \subseteq I \subseteq W$. Since p is a maximal ideal, either $I = p$ and $H = pG$, or $I = W$ and $H = G$. Thus G/pG has no proper submodules so p-rank $G = \operatorname{length} G/pG = 1$.

($\Rightarrow$): If H is quasi-equal to G then $wH \subseteq G$ for some $w \neq 0 \in W$, and if $wH = IG$ then $H = w^{-1}IG$. It thus suffices to consider the case where $H \subseteq G$ and H is quasi-equal to G. Then by Proposition 3.11 G/H is a finite length homomorphic image of G, hence by Proposition 3.33 $G/H \approx W/I$ for some ideal I. Then $I(G/H) = 0$ so $IG \subseteq H \subseteq G$. But by Proposition 3.33 again $G/IG \approx W/J$ for some ideal J and since $I(W/J) = 0$ we get $I \subseteq J$. Thus $\operatorname{length} G/IG = \operatorname{length} W/J \leq \operatorname{length} W/I = \operatorname{length} G/H \leq \operatorname{length} G/IG$ since $IG \subseteq H$. Thus $H = IG$. $\boxed{\checkmark}$

Recall that we say that a submodule H of G is **fully invariant** if $\varphi(H) \subseteq H$ for all $\varphi \in \operatorname{End} H$.

PROPOSITION 3.36. *A Murley module G is a direct sum of strongly indecomposable modules. In particular, an indecomposable Murley module is strongly indecomposable. Furthermore every summand of G is fully invariant.*

PROOF: We know that any finite rank torsion free module G is quasi-equal to a direct sum $H_1 \oplus \cdots \oplus H_n$, where the H_i are strongly indecomposable. If G is a Murley module then by Proposition 3.35 $G = IH_1 \oplus \cdots \oplus IH_n$ for some fractional ideal I. Since the IH_k are strongly indecomposable by Proposition 3.10, the result is established.

Now if $G = H \oplus K$ and if $\varphi\colon H \to K$ then $\varphi(H)_*$ is p-divisible by every prime such that K is p-divisible, and also by every prime such that H is p-divisible. But for every prime p either H or K is p-divisible since p-rank$(H \oplus K) \leq 1$. Thus $\varphi(H)_*$ is divisible, and since G is reduced $\varphi(H)_* = 0$, so $\varphi = 0$. Thus $\operatorname{Hom}(H, K) = 0$ and likewise $\operatorname{Hom}(K, H) = 0$. It follows immediately that H and K are fully invariant in G. $\boxed{\checkmark}$

Direct sum decompositions involving fully invariant summands are extremely nice. They are discussed at the beginning of Chapter 11.

COROLLARY 3.37. *If G is a Murley module and $G = \bigoplus G_i$, then $\operatorname{End} G \approx \prod \operatorname{End} G_i$.*

PROOF: Immediate since the G_i are fully invariant submodules of G. $\boxed{\checkmark}$

COROLLARY 3.38. *The class of Murley modules has the Krull-Schmidt Property. In fact, if G is a Murley module and $G = G_1 \oplus \cdots \oplus G_s = H_1 \oplus \cdots \oplus H_t$, where the G_i and H_i are indecomposable, then $s = t$ and the H_i can be renumbered so that for each i, G_i and H_i are identical submodules of G.*

PROOF: This follows from the fact that the G_i and H_i are fully invariant submodules of G. If γ_1 is the projection of G onto G_1 then for each i, $\gamma_1(H_i) \subseteq H_i$ because H_i is fully invariant. It follows that γ_1 restricts to a projection from H_i onto $H_i \cap G_1$. Since H_i is indecomposable, either $H_i = G_1$ or $H_i = 0$. We can then renumber the H_i so that $H_1 = G_1$ and complete the proof by induction. $\boxed{\checkmark}$

PROPOSITION 3.39. *Let H be a p-local Murley module and G a reduced module containing H as an essential submodule. Then H is quasi-equal to G.*

PROOF: G/H is a proper submodule of QG/H because G is reduced. But since H is a p-local Murley module, by Proposition 1.24 $QG/H = QH/H \approx W(p^\infty)$. Therefore all proper submodules of QG/H are cyclic, so G/H is cyclic and H is quasi-equal to G. $\boxed{\checkmark}$

MURLEY RINGS.

PROPOSITION 3.40. *Let R be a Murley ring and S a subring of QR quasi-equal to R. Then $R = S$.*

PROOF: By Proposition 3.35 $S = IR$ for some fractional ideal $I \subseteq Q$. Thus S is an R-module. Since $1 \in S$, it follows that $R \subseteq S$. Analogously, $S \subseteq R$. $\boxed{\checkmark}$

PROPOSITION 3.41. *Let R and S be finite rank torsion free Murley rings such that* $\text{p-rank}\, R = \text{p-rank}\, S$ *for all p and such that R is an essential subring of S. Then $R = S$.*

PROOF: It suffices to prove that $R_p = S_p$ for all p. Therefore we may assume that W is local. It now follows from Proposition 3.39 that R is quasi-equal to S and therefore $R = S$ by Proposition 3.40. $\boxed{\checkmark}$

LEMMA 3.42. *If R is a Murley ring and L, M are R-modules which are torsion free as W-modules* THEN

1. *If R is not p-divisible then pR is a prime ideal in R and* $(pR)\text{-rank}\, M = \text{p-rank}\, M$.
2. $\text{p-rank}(L \otimes_R M) = (\text{p-rank}\, L)(\text{p-rank}\, M)$.
3. *If furthermore L if finitely generated over R and M is reduced (as a W-module) then $L \otimes_R M \approx L \star M$.*
4. *If M is a reduced R-module then $R \star M \approx M$.*

PROOF: (1) First note that if R is not p-divisible then $\text{p-rank}\, R = 1$ and so $R/pR \approx W/p$ and hence R/pR is a simple W-module and thus is a fortiori simple as an R-module. Thus pR is a maximal ideal of R. Now $(pR)\text{-rank}\, M = \operatorname{length}_R M/pRM = \operatorname{length}_R M/pM = \dim_{R/pR} M/pM = \dim_{W/p} M/pM = \text{p-rank}\, M$ since $R/pR \approx W/p$.

(2) If X is an R-module then $\text{p-rank}\, X = \operatorname{length} X/pX = \operatorname{length} R/pR \otimes_R X$. (The length here is computed over W.) Also $R/pR \otimes_R R/pR \approx (R/pR)/p(R/pR) \approx R/pR$. From this it follows that $\text{p-rank}\, L \otimes_R M = \operatorname{length} R/pR \otimes_R L \otimes_R M = \operatorname{length} L/pL \otimes_R M/pM$. On the other hand, by Proposition 1.36 $(\text{p-rank}\, L)(\text{p-rank}\, M) = \text{p-rank}\, L \otimes M = \operatorname{length} L/pL \otimes_W M/pM$. Thus it suffices to see that $L/pL \otimes_R M/pM \approx L/pL \otimes_W M/pM$. Now if $\text{p-rank}\, R = 0$ then the R-modules L and M are p-divisible and all the p-ranks are trivial. Otherwise $\text{p-rank}\, R = 1$. Since L/pL and M/pM are vector spaces over R/pR it suffices to see that $(R/pR) \otimes_R (R/pR) \approx (R/pR) \otimes_W (R/pR)$, in other words that $(R/pR) \otimes_W (R/pR) \approx R/pR$. But this is clear since $R/pR \approx W/pW$.

(3) If L is finitely generated over R and M is reduced, then $L \otimes_R M$ is reduced. This is clear is L is free, or even if L is projective, and this case will provide as much as we need to establish the propositions which follow. Once these are established and we know that R is a finite product of dedekind domains, it will follow that any finitely generated torsion free L is projective so that this special case is in fact not special at all.

Assuming then that $L \otimes_R M$ is reduced, the surjection $L \otimes M \to L \otimes_R M$ induces a surjection $\mu\colon L \star M \to L \otimes_R M$. Since $\text{p-rank}(L \star M) = (\text{p-rank}\, L)(\text{p-rank}\, M) = \text{p-rank}\, L \otimes_R M$, $\operatorname{Ker} \mu$ has p-rank 0 for all p. Thus $\operatorname{Ker} \mu$ is divisible, and since $L \star M$ is reduced $\operatorname{Ker} \mu = 0$ and μ is an isomorphism.

(4): By (2) $R \star M \approx R \otimes_R M \approx M$. $\boxed{\checkmark}$

PROPOSITION 3.43. *Let R be a Murley ring.*

1. $R \star R \approx R$.
2. $\operatorname{End} R \approx R$.
3. *R is commutative.*
4. *R has trivial nil radical.*
5. *If L and M are R-modules and M is reduced, then every homomorphism $L \to M$ is R-linear.*

PROOF: (1) Immediate from Lemma 3.42.

(5) If $\varphi\colon L \to M$ then φ induces an R-linear map $R \star L \to R \star M$. But by Lemma 3.42 $R \star L$ and $R \star M$ are just L and M.

(2) By (5) $\operatorname{End} R = \operatorname{End}_R R \approx R$.

(3) If $r \in R$ then there is a map $\varphi \in \operatorname{End} R$ defined by $\varphi(x) = rx$. By (5) φ is R-linear. Thus for all $x, r \in R$, $rx = \varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = xr$.

(4) It suffices to show that if $r \in R$ with $r^2 = 0$ then $r = 0$. If $r^2 = 0$ then the multiplication map $x \mapsto xr$ gives a short exact sequence

$$0 \to \operatorname{ann}(r) \xrightarrow{\subseteq} R \to rR \to 0$$

so that for any p, $\text{p-rank}\, R = \text{p-rank}\, rR + \text{p-rank}\, \text{ann}(r)$. Let $I = (rR)_*$. Now if R is p-divisible then I is also p-divisible. Likewise if rR is p-divisible then the pure ideal I is also p-divisible. And if rR is not p-divisible then $\text{p-rank}\, rR = \text{p-rank}\, R = 1$ and $\text{p-rank}\, \text{ann}(r) = 0$. But $rR \subseteq \text{ann}(r)$ since $r^2 = 0$ and so $I = (rR)_* \subseteq \text{ann}(r)$ since $\text{ann}(r)$ is pure in R, and so in this case also $\text{p-rank}\, I = 0$. Thus I is p-divisible for every prime p, so I is divisible. Since R is reduced, $I = 0$. Thus $rR = 0$, i.e. $r = 0$. $\boxed{\surd}$

PROPOSITION 3.44. *A Murley ring has no proper idempotents if and only if it is strongly indecomposable as a W-module.*

PROOF: ($\Leftarrow$): If r is an idempotent in R then left multiplication by r is an idempotent endomorphism of R, so R is not indecomposable, hence certainly not strongly indecomposable.

($\Rightarrow$): Suppose that R_1 and R_2 are W-submodules of R such that $R = R_1 \oplus R_2$. By Proposition 3.36 R_1 and R_2 are fully invariant in R. In particular, they are closed under multiplication by elements of R, i.e. they are ideals. Thus R is the ring-theoretic product $R_1 \times R_2$. Since R has no proper idempotents, it follows that $R_1 = 0$ or $R_2 = 0$. Thus R is an indecomposable W-module. By Proposition 3.36 R is strongly indecomposable. $\boxed{\surd}$

PROPOSITION 3.45. *Let R be a Murley ring with no non-trivial idempotents. If R has infinite rank assume also that that QR is a field. Then R is a dedekind domain and every ideal $I \subseteq R$ has the form $\mathfrak{a}R$, where $\mathfrak{a} = W \cap I$.*

PROOF: By Proposition 3.43 R is commutative with trivial nil radical. Thus if $\text{rank}\, R < \infty$ then QR is a finite dimensional commutative Q-algebra with trivial radical, hence a product of fields. By Proposition 3.44 R is strongly indecomposable, therefore by Proposition 3.16 $\text{QEnd}\, R$ has no proper idempotents. Then QR has no proper idempotents, since QR can be identified as a subring of $\text{QEnd}\, R$. Therefore QR is in fact a field.

Now let I be a non-trivial ideal in R. Since QR is a field, $IQR = QR$. But QR is the localization of R with respect to the non-zero elements of W, so from the fact that $1 \in IQR$ we conclude that in IQR, $1 = i/s$ for some $i \in I$, $s \neq 0 \in W$. Thus $i \in W \cap I$ so $W \cap I \neq 0$. If $w \neq 0 \in I \cap W$ then $wR \subseteq IR = I$, showing that I is quasi-equal to R. Then by Proposition 3.35 $I = \mathfrak{a}R$ for some ideal $\mathfrak{a}$ of W. Then $\mathfrak{a} \subseteq I \cap W$ so that $I = \mathfrak{a}R \subseteq (I \cap W)R \subseteq IR = I$, so that we may choose $\mathfrak{a} = I \cap W$.

Now if $\mathfrak{a}R$ is a non-trivial ideal in R, where $\mathfrak{a} \subseteq W$, then $\mathfrak{a}^{-1}R$ is an R-submodule of QR and $\mathfrak{a}^{-1}R\mathfrak{a}R = \mathfrak{a}^{-1}\mathfrak{a}R = R$. Thus every non-trivial ideal in R is invertible, so R is a dedekind domain. $\boxed{\surd}$

THEOREM 3.46. *A finite rank Murley ring is a finite product of dedekind domains. In particular, a Murley ring is indecomposable if and only if it is a dedekind domain.*

PROOF: By Corollary 3.36 R is the direct sum of strongly indecomposable modules $R_1, \ldots, R_n$ and these are fully invariant in R. But this implies that they are ideals in R, so R is the ring-theoretic product of $R_1, \ldots, R_n$. These R_i must then all be indecomposable Murley rings and therefore by Proposition 3.44 and Proposition 3.45 they are dedekind domains. $\boxed{\checkmark}$

THE TORSION SUBMODULE OF EXT. Finally, for those who know about Ext, we give the following important result due to Carol Walker.

THEOREM 3.47. *Let* $\mathcal{E}\colon 0 \to H \to G \to K \to 0$ *be a short exact sequence. Then this sequence is quasi-split if and only if the corresponding element of* $\mathrm{Ext}(K, H)$ *lies in the torsion submodule.*

PROOF: For $w \neq 0 \in W$ multiplication by w on K is monic, hence multiplication by w on $\mathrm{Ext}(K, H)$ is surjective since $\mathrm{Ext}(\_\,, H)$ is right exact. I.e. $\mathrm{Ext}(K, H)$ is divisible. It follows that $Q \otimes \mathrm{Ext}(K, H)$ equals the quotient of $\mathrm{Ext}(K, H)$ modulo its torsion submodule. Now since Q is flat the short exact sequence $\mathcal{E}\colon 0 \to H \to G \to K \to 0$ leads to the long exact sequence $0 \to \mathrm{QHom}(K, H) \to \mathrm{QHom}(K, G) \to \mathrm{QHom}(K, K) \to Q \otimes \mathrm{Ext}(K, H) \to Q \otimes \mathrm{Ext}(K, G) \to Q \otimes \mathrm{Ext}(K, K) \to 0$. Now if the original sequence is quasi-split, then $\mathrm{QHom}(K, G) \to \mathrm{QHom}(K, K)$ is surjective, and conversely if this map is surjective then $1_K \in \mathrm{QHom}(K, K)$ lifts to a map in $\mathrm{QHom}(K, G)$, which is equivalent to saying that the original sequence quasi-splits. Thus we see that the original sequence quasi-splits if and only if the image of $1_K \in \mathrm{QHom}(K, K)$ in $Q \otimes \mathrm{Ext}(K, H)$ is trivial, i.e. if the connecting homomorphism takes 1_K into the torsion submodule of $\mathrm{Ext}(K, H)$. But it is well known in homological algebra [see Fuchs I] that the image of 1_K in $\mathrm{Ext}(K, H)$ is the element corresponding to the original sequence $\mathcal{E}$. $\boxed{\checkmark}$

The following proposition gives an alternative, more explicit proof of the above theorem.

PROPOSITION 3.48. *Let* $H \lhd G$ *and let* $\mathcal{E}$ *be the element of* $\mathrm{Ext}(G/H, H)$ *represented by the short exact sequence* $0 \to H \to G \xrightarrow{\gamma} G/H \to 0$. *Let* $w \neq 0 \in W$. *Then* $w\mathcal{E} = 0$ *if and only if* H *is a direct summand of* $H + wG$. *Furthermore this is equivalent to the existence of* $X \subseteq G$ *with* $X \cap H = 0$ *such that* $wG \subseteq H \oplus X \subseteq G$ *(which implies, of course, that* $\mathcal{E}$ *is quasi-split).*

PROOF: Multiplication by w gives a map $G/H \to G/H$ which induces the map $\mathrm{Ext}(H, G/H) \to \mathrm{Ext}(H, G/H)$ taking $\mathcal{E}$ to $w\mathcal{E}$. But $G/H \xrightarrow{w} G/H$ factors as $G/H \xrightarrow{w} w(G/H) \xrightarrow{\subseteq} G/H$ where the first map is an isomorphism, so the extension $w\mathcal{E}$ is isomorphic to the extension $\mathcal{E}'$ which is the pull-back of $\mathcal{E}$ under the inclusion $w(G/H) \hookrightarrow G/H$. This pull-back is shown in the following diagram:

$$\begin{tikzcd}
\mathcal{E}'\colon 0 \arrow[r] & H \arrow[r] \arrow[d, equal] & H + wG \arrow[r] \arrow[d, "\subseteq"'] & w(G/H) \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
\mathcal{E}\colon 0 \arrow[r] & H \arrow[r, "\subseteq"] & G \arrow[r] & G/H \arrow[r] & 0 .
\end{tikzcd}$$

(In fact, if the right hand square is to be a pull-back, then the middle module in the top sequence must be simply the inverse image of $w(G/H)$ in G, i.e. $H + wG$, as shown.)

Then $w\mathcal{E} = 0$ if and only if the top sequence splits, i.e. if and only if H is a direct summand of $H + wG$.

By Proposition 3.18 this is equivalent to the existence of $X \subseteq G$ with $H + wG = H \oplus X$ and $wG \subseteq H + wG = H \oplus X \subseteq G$. $\boxed{\checkmark}$

RESTATEMENT. If $H \lhd G$ then the element of $\operatorname{Ext}(K, H)$ corresponding to the short exact sequence $0 \to H \to G \to G/H \to 0$ is annihilated by an element $w \neq 0 \in W$ if and only if there exists $X \subseteq G$ with $X \cap H = 0$ such that G is generated by $H \oplus X$ together with a family $\{(h_i + k_i)/w\}_{i \in I}$ with $w \neq 0 \in W$ and $h_i \in H$, $k_i \in X$. Furthermore in this case there is no loss of generality in assuming that the index set I is finite.

We can (unhistorically) think of the motivation for inventing the category of finite rank torsion free modules under quasi-homomorphisms as follows: Finite rank torsion free modules are too complicated to be understood simply by looking at them. Attempts to understand these objects will consist of finding ways to associate with a module G various other objects which are sufficiently simpler than G that we have some reasonable chance of comprehending them, and yet still provide useful information about the original module G. For instance, we can associate to G various numbers, such as its rank and p-rank. Or sets, such as $\mathbf{T}(G)$ and $\mathbf{CT}(G)$.

A very natural approach is to find functors from the category of finite rank torsion free W-modules into nicer categories. For instance one has a functor into the category of finite dimensional vector spaces by associating to each G its divisible hull QG. This takes us into a category that has extremely nice properties, but the price we pay is that the only information QG provides about G is its rank.

The category of finite length modules over a finite dimensional Q-algebra Λ is a lot nicer than the category of finite rank torsion free W-modules, even though it still has its complications. Alas, finding a useful functor into this category is not feasible. But if we examine what it is that makes this category nice, it is largely the fact that the homomorphism groups between two finite length Λ-modules are finite dimensional vector spaces, and in particular that the endomorphism ring of a finite length module is a finite dimensional algebra. This is what makes the Krull-Schmidt-Azumaya Theorem applicable to the category.

The category of finite rank torsion free W-modules under quasi-homomorphisms gives us a category which has the nice properties of the category of finite length modules over a finite dimensional algebra, and at the same time is sufficiently complicated that much of the important information about a finite rank torsion free module persists when we look at it in this category.

It is clear that if G is quasi-equal to G' and H quasi-equal to H', then $\operatorname{QHom}(G, H) = \operatorname{QHom}(G', H')$. For this reason, there is a lot to be said for thinking of the objects in the category of finite rank torsion free modules under quasi-homomorphisms as being not modules, but quasi-equality classes of modules. Certainly *this makes sense if one remembers that from the point of view of category theory objects exist merely as place-holders for morphisms.*

This way of thinking can lead to some valuable insights. For instance in the category of modules under quasi-homomorphisms W *is a simple object: its only subobjects are the ideals, and these are all quasi-equal to* W*. Furthermore if* W *is a local ring, then in the category of modules under quasi-homomorphisms it can be seen that all modules have finite length. In fact, since any finite rank module has a filtration where the factors are rank-one modules, it suffices to see that rank-one modules have finite length. But from the point of view of quasi-equality there are only two submodules of* Q*, namely* W *and* Q*. This insight was what led your author to suspect that some of the work which Maurice Auslander and Idun Reiten were doing for abelian categories where all objects have finite length might be applicable to the study of finite rank torsion free modules.*

On the other hand, purely as a matter of style it is certainly more convenient to talk about modules than to talk about quasi-equality classes, and so that is the language in which the discussion in this book will take place.