Convergence of Infinite Series in General and Taylor Series in Particular

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Some Series Converge: The Ruler Series

At first, it doesn't seem that it would ever make any sense to add up an infinite number of things. It seems that any time one tried to do this, the answer would always be infinitely large.

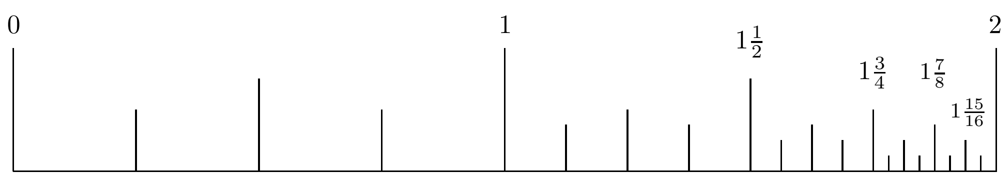
The easiest example that shows that this need not be true is the series I like to call the "Ruler Series:"

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

It should be clear here what the "etc etc" (...) at the end indicates, but the k^{th} term being added here (if one counts 1 as the 0^{th} term and 1/2 as the 1^{st}, etc.) is $1/2^k$. For instance the 10th term is $1/2^{10}=1/1024$. If one looks at the sums as one takes more and more terms of these series, the sum is 1 if one takes only the first term, $1\frac{1}{2}$ if one takes the first two, $1\frac{3}{4}$ if one adds the first three terms of the series. As one adds more and more terms, one gets a sequence of sums

$$1 \qquad 1\tfrac{1}{2} \qquad 1\tfrac{3}{4} \qquad 1\tfrac{7}{8} \qquad 1\tfrac{15}{16} \qquad 1\tfrac{31}{32} \qquad 1\tfrac{63}{64} \qquad \cdots$$

These numbers are the ones found on a ruler as one goes up the scale from 1 towards 2, each time moving towards the next-smaller notch on the ruler.



Once one sees the pattern, two things are clear:

(1) Even if one adds an incredibly large number of terms in this series, the sum never gets larger than 2.

(2) By adding enough terms, the sum can be made arbitrarily close to 2.

We say that the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

converges to 2. Symbolically, we indicate this by writing

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=2\,.$$

This notation doesn't make any sense if interpreted literally, but it is common for students (and even many teachers) to interpret this as meaning "*If one could* add all the infinitely many terms, then the final sum would be 2." This, unfortunately, is not too much different from saying, "*If* horses could fly then riders could chase clouds." The fact is that horses cannot fly and one cannot add together an infinite number of things. Instead, one is taking the *limit* as one adds more and more and more of the terms in the series.

The fact that one is taking a limit rather than adding an infinite number of things may seem like a fine point that only mathematicians would be concerned with. However certain things happen with infinite series that will seem bizarre unless you remember that one is not actually adding together all the terms.

Some Series Diverge: The Harmonic Series

The nature of the human mind seems to be that we assume that the particular represents the universal. In other words, in this particular instance, from the fact that the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

converges, one is likely to erroneously infer that *all* infinite series converge. This is clearly not the case. For instance,

$$1+2+3+4+\cdots$$

is an infinite series that clearly cannot converge. For that matter, the series

$$1+1+1+1+1+\cdots$$

also does not converge.

These examples illustrate a rather obvious rule: *An infinite series cannot converge unless the terms eventually get arbitrarily small.* In more formal language:

> An infinite series $a_0+a_1+a_2+a_3+\cdots$ cannot converge unless $\lim_{k\to\infty} a_k = 0$.

The natural mistake to make now is to assume that any infinite series where $\lim_{k\to\infty} a_k = 0$ will converge. Remarkably enough, this is not true.

For instance, consider the following series:

$$1 + \frac{1}{2} \; + \frac{1}{4} + \frac{1}{4} \; + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \; + \frac{1}{16} + \cdots + \frac{1}{16} \; + \frac{1}{32} + \cdots .$$

The idea is that there will be 8 terms equal to $\frac{1}{16}$, 16 terms equal to $\frac{1}{32}$, 32 terms equal to $\frac{1}{64}$, etc. The k^{th} term here (if we count 1 as the first) is $1/\gamma(k)$, where we define $\gamma(k)$ to be the smallest power of 2 which is greater than or equal to k:

$$\begin{aligned}
&\gamma(2) = 2 \\
&\gamma(3) = \gamma(4) = 4 \\
&\gamma(5) = \cdots = \gamma(8) = 8 \\
&\gamma(9) = \cdots = \gamma(16) = 16 \\
&\gamma(17) = \cdots = \gamma(32) = 32 \\
&\text{etc.}
\end{aligned}$$

(For a more formulaic definition, we can define $\gamma(k) = 2^{\ell(k)}$ with $\ell(k) = \lceil \log_2 k \rceil$, where $\lceil x \rceil$ is the *ceiling* of x: the smallest integer greater than or equal to x. For instance, since $2^4 < 23 < 2^5$, it follows that $\ln_2 23 = 4.***\ldots$ and so $\ell(23) = \lceil 4.*** \rceil = 5$ and so $\gamma(23) = 2^5 = 32$.)

Clearly in this series, $\lim_{k\to\infty} a_k = 0$. On the other hand, we can see that the second term of the series is $\frac{1}{2}$, and the sum of the third and fourth terms is also $\frac{1}{2}$, and so is the sum of the fourth through eight terms. The ninth through sixteenth terms also add up to $\frac{1}{2}$, as do the seventeenth through the thirty-secoond:

$$\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32} = \frac{1}{2}.$$

We can see the pattern easily by inserting parenthesis into the series:

$$1 + \frac{1}{2} \; + (\frac{1}{4} + \frac{1}{4}) \; + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) \; + (\frac{1}{16} + \cdots + \frac{1}{16}) \; + (\frac{1}{32} + \cdots + \frac{1}{32}) + \cdots .$$

The terms within each set of parentheses add up to $\frac{1}{2}$. Thus as one goes further down the series, one keeps adding a new summand of $\frac{1}{2}$ over and over again.

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .$$

Thus, by including enough terms, one can make the partial sum of this series as large as one wishes. Hence the series

$$1 + \frac{1}{\gamma(2)} + \frac{1}{\gamma(3)} + \frac{1}{\gamma(4)} + \frac{1}{\gamma(5)} + \frac{1}{\gamma(6)} + \cdots$$

does not converge.

This example may not seem very profound, but by using it, it is easy to see that the **Harmonic Series**

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \cdots$$

also **does not converge.** Despite the fact that the terms one is adding one keep getting smaller and smaller, to the extent that eventually they fall below the level where a calculator can keep track of them, nonetheless if one takes a sufficient number of terms amd keeps track of all the decimal places, the sum can be made arbitrarily huge.

In fact, the k^{th} term of the Harmonic Series is $\dfrac{1}{k}$. If $\gamma(k)$ is the function we defined above, then by definition $k \le \gamma(k)$. Thus

$$\frac{1}{k} \;\ge\; \frac{1}{\gamma(k)}\,.$$

Thus the partial sums of the Harmonic Series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots$$

are even larger than the partial sums of the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\cdots+\frac{1}{\gamma(k)}+\cdots$$

which, as we have already seen, does not converge. Therefore the Harmonic Series must also not converge.

In fact, if we use parentheses to group the Harmonic Series we get

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\left(\frac{1}{17}+\cdots+\frac{1}{32}\right)+\cdots,$$

and we can see that the group of terms within each parenthesis adds up to a sum greater than $\frac{1}{2}$, making it clear that if one takes enough terms of the Harmonic Series one can get an arbitrarily large sum. (The parentheses here do not change the series at all; they only change the way we look at it.)

The Geometric Series

The Ruler Series can be rewritten as follows:

$$1+(\tfrac{1}{2})+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3+(\tfrac{1}{2})^4+\ldots$$

This is an example of a *Geometric Series:*

$$1+x+x^2+x^3+x^4+x^5+x^6+\cdots$$

If $-1<x<1$, then we will see that this series converges to $1/(1-x)$. On the other hand, if $x\ge 1$ or $x\le -1$ then the series diverges.

The second of these assertions is easy to understand. If $x=1$, for instance, then the Geometric series looks like

$$1+1+1+1+1+\cdots$$

and obviously does not converge. Making x larger can only make the situation worse.

On the other hand, if $x=-1$ then the series looks like

$$1-1+1-1+1-1+1-\cdots$$

Even though the partial sums of this series never get any larger than 1 or more negative than 0, the series doesn't converge since the partial sums keep jumping back and forth between 0 and 1. Making x more negative can only make the situation worse. For instance, when $x = -2$ we get

$$1 - 2 + 4 - 8 + \cdots \pm 2^k \cdots$$

which clearly does not converge.

To understand what happens when $|x| < 1$, we need a factorization formula from college algebra:

$$(1-x)(1 + x + x^2 + x^3 + \cdots + x^n) = 1 - x^{n+1}$$

Thus

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$

Thus if we take the displayed formula above and take the limit as n approaches $+\infty$, we get

$$1 + x + x^2 + x^3 + x^4 + \cdots = \lim_{n\to\infty} \frac{1}{1-x} - \lim_{n\to\infty} \frac{x^n}{1-x}.$$

It's important to note that when one takes this limit, x does not change; only n changes. It's also important to know that if $|x| < 1$, then x^n converges to 0 as n goes to ∞. (A calculator will show that this happens even if x is very close to 1, say $x = .9978$.) Thus if $|x| < 1$ then on the right-hand side we get

$$\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} - \frac{1}{1-x} \lim_{n\to\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1.}$$

The geometric series is of crucial important in the theory of infinite series. Most of what is known about the convergence of infinite series is known by relating other series to the geometric series.

By using some simple variations, we can get a number of different series from the geometric series. For instance the series

$$1 + 3x + 9x^2 + 27x^3 + 81x^4 + \cdots + 3^k x^k + \cdots$$

can be rewritten as

$$1 + (3x) + (3x)^2 + (3x)^3 + \cdots + (3x)^k + \cdots$$

which is just the geometric series with $3x$ substituted for x. Thus from the formula for the geometric series, we get

$$1 + 3x + 9x^2 + 27x^3 + 81x^4 + \cdots + 3^k x^k + \cdots = \frac{1}{1-3x}.$$

This will converge when $-1 < 3x < 1$, i.e. when $-\frac{1}{3} < x < \frac{1}{3}$.

Likewise, if we look at

$$x^2 + x^3 + x^4 + \cdots$$

we can factor out the x^2 to see that

$$\begin{aligned} x^2 + x^3 + x^4 + \cdots &= x^2(1 + x + x^2 + x^3 + \cdots) \\ &= x^2\left(\frac{1}{1-x}\right) = \frac{x^2}{1-x}\,. \end{aligned}$$

It converges for $-1 < x < 1$.

For a more complicated example, consider

$$5x^3 + 10x^5 + 20x^7 + 40x^9 + \cdots + 5 \cdot 2^k x^{2k+3} + \cdots$$

We can evaluate this as

$$\begin{aligned} 5x^3 + 10x^5 + 20x^7 + 40x^9 + \cdots &= 5x^3\,[1 + 2x^2 + (2x^2)^2 + (2x^2)^3 + \cdots + (2x^2)^k + \cdots] \\ &= \frac{5x^3}{1-2x^2}\,, \end{aligned}$$

which converges when $|\,2x^2| < 1$, i. e. $|\,x| < 1/\sqrt{2}$.

For still another trick, consider the function

$$f(x) = \frac{1}{2x+3}\,.$$

This can be expanded in a variation of the geometric series as follows:

$$\begin{aligned} f(x) = \frac{1}{2x+3} &= \frac{1}{3} \cdot \frac{1}{1-(-\frac{2}{3}x)} \\ &= \frac{1}{3}\left(1 + (-\tfrac{2}{3}x) + (-\tfrac{2}{3}x)^2 + (-\tfrac{2}{3}x)^3 + \cdots\right) \\ &= \frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81} + \cdots + (-1)^n \frac{2^n x^n}{3^{n+1}} + \cdots\,. \end{aligned}$$

This converges when $\left|\dfrac{2x}{3}\right| < 1$, i. e. when $-\dfrac{3}{2} < x < \dfrac{3}{2}$.

Now consider $f(x) = 1/(x^2 - 2x + 8)$. Using partial fractions, we can expand this as

$$\begin{aligned} f(x) &= \frac{1}{(x-2)(x-4)} = \frac{\frac{1}{2}}{x-4} - \frac{\frac{1}{2}}{x-2} = \frac{\frac{1}{2}}{2-x} - \frac{\frac{1}{2}}{4-x} \\ &= \frac{1}{4} \cdot \frac{1}{1-\frac{x}{2}} \quad - \quad \frac{1}{8} \cdot \frac{1}{1-\frac{x}{4}} \\ &= \left(\frac{1}{4} + \frac{x}{8} + \frac{x^2}{16} + \frac{x^3}{2^5} + \cdots \right) \\ &\qquad - \left(\frac{1}{8} + \frac{x}{32} + \frac{x^2}{2 \cdot 4^3} + \frac{x^3}{2 \cdot 4^4} + \cdots \right) \\ &= \frac{1}{4} + \frac{3x}{16} + \frac{7x^2}{64} + \cdots + \frac{(2^{n+1}-1)x^n}{2^{2n+3}} + \cdots . \end{aligned}$$

For this to converge, we need both $\left|\frac{x}{2}\right| < 1$ and $\left|\frac{x}{4}\right| < 1$. Thus the series converges for $-2 < x < 2$.

Note. We will see below that subtracting two series in this way can sometimes be a little more delicate than one might think. It's generally important to interlace the terms of the two series in such a way that the balance between positive terms and negative terms is not affected. For these particular series, however, this is not an issue in any case because, using a concept to be defined below, these series **converge absolutely** for any value of x for which they converge at all.

If $q(x)$ is a polynomial with a repeated factor in the denominator, then this partial fractions trick cannot be used this simply to expand a function $p(x)/q(x)$ (where $p(x)$ is a polynomial with no factor in common with $q(x)$). To expand a function like $1/(x-5)^3$, for instance, one needs the *Negative Binomial Series*, discussed below.

Positive Series

When one thinks of a series diverging, one usually thinks of one like the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

that just keeps getting larger and larger, and can in fact be made as large as one wants by taking enough terms. In symbolic form, one represents this behavior by writing

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

However, a series can fail to converge in a less obvious way. For instance,

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \cdots .$$

This series is not very subtle, but it illustrates the point. The partial sums oscillate between 0 and 1, and thus never stay close to any limit.

However when all the terms of a series are positive, then this kind of wavering-back-and-forth behavior cannot occur, since as we add on more and more terms the sums keep getting larger and larger.

In fact, the following is true:

> If all the terms of a series are positive, then either the series converges or $\sum_{n=1}^{\infty} a_n = +\infty$.

(If, on the other hand, all the terms are negative, then either the series converges or $\sum_{n=1}^{\infty} a_n = -\infty$. It is only series having both positive and negative terms that can oscillate.)

Before suggesting why the above principle should be accepted, I want to state it in its more usual form.

Bounded Series. A number B is said to be an **upper bound** for the series if $\sum_1^N a_n \le B$, no matter how many terms we take. (If a series has one upper bound, then it has lots of them, since $B+1$ would also be an upper bound, as would $B+8$, $B+\pi$, etc.) A series that has an upper bound is called **bounded above**.

We have agreed to write $\sum_{n=1}^{\infty} a_n = \infty$ to indicate that the sums $\sum_{n=1}^{N} a_n$ can be made arbitrarily large by taking N large enough. This is precisely the same as saying that the series is *not* bounded above. Therefore, the principle stated above can be rephrased as saying that the only possibility for a positive series are either to converge or to be *not* bounded above.

In other words, the principle in question says that

> If a positive series is bounded above, then it converges.

Although this principle is usually stated as an axiom in most books, I would like to give some indication as to why it should be believed.

Consider the integer part of the sums as we take more and more terms of a positive series. (These sums are usually called the **partial sums** of the series.)

For instance there could be a series where the partial sums are as follows:

Partial Sum	Integer Part of Partial Sum
$a_1 = 4.9873$	4
$a_1 + a_2 + a_3 + a_4 = 6.594$	6
$a_1 + \cdots + a_{20} = 8.27$	8
$a_1 + \cdots + a_{100} = 8.316$	8
$a_1 + \cdots + a_{600} = 8.359$	8

If the series is bounded, then this sequence of integers can't keep getting larger and larger. On the other hand, they can never get smaller if the series is positive, since the partial sums keep getting larger. Therefore, eventually these integer parts must stabilize. (For instance in the example above, it certainly looks as though the integer parts of the partial sums stabilize at a value of 8.)

The point is that the partial sums as a whole can keep getting bigger and bigger forever, by smaller and smaller increments, and still be bounded above. But *integers* can't do this. If a sequence of *integers* never decreases and never gets larger than a certain upper bound, then eventually it has to become constant.

At the risk of running this into the ground, let me explain in even more detail. Suppose we have a positive series $\sum b_n$ and we know that it is bounded above. This means that there is some number, maybe 43.17, that $b_1 + \cdots + b_n$ never gets any bigger than. I. e. 43.17 is an **upper bound** for the partial sums. Now lets say that $b_1 = -12.642$. Now we can look at the integers between -13 and 44. Some of these (44, for instance) are upper bounds for the partial sums, and some of them (-13, for instance) are not.

So somewhere between -13 and 43 there is an integer K so that $b_1 + \cdots + b_n < K + 1$ no matter how large n is, but $b_1 + \cdots + b_n \geq K$ for some value of n (and therefore for all succeeding values of n as well, since the series is positive).

For instance, in the example above, we can see that $a_1 + \cdots + a_{20} \geq 8$, but it certainly looks like $a_1 + \cdots + a_n < 9$ for all n. (We can't say for sure, though, without knowing the whole series.) Assuming that this is so, then the integer part of $a_1 + \cdots + a_n$ eventually stabilizes at 8.

Now consider the first decimal to the right of the decimal point for the partial sums. At first, this decimal may waver back and forth in a rather erratic fashion (for instance, in the example above we have 4.**9**873, 6.**5**94, 8.**2**73 as the first, fourth, and twentieth partial sums). But once the integer part of the partial sum stabilizes, then first digit to the right of the decimal point can't decrease, since the partial sums are increasing. Since there are only ten choices (0 — 9) for this digit, eventually it has to also stabilize.

For instance, we might imagine the example above (which converges much much more slowly than most series one works with) continuing as follows.

Partial Sum	Initial Two Digits
$a_1 + \cdots + a_{100} = 8.316$	8.3
$a_1 + \cdots + a_{600} = 8.359$	8.3
$a_1 + \cdots + a_{1000} = 8.364$	8.3

It certainly seems as if the initial two digits will be 8.3 from the 100^{th} term on out. (Again, though, one can't be absolutely sure without seeing the whole series.)

Notice also that once the tenths digit stabilizes, the hundredths digit (the second digit to the right of the decimal point) cannot decrease. Therefore eventually the hundredths digit must stabilize as well. The sequence of partial sums might continue

Partial Sum	Initial Three Digits
$a_1 + \cdots + a_{1000} = 8.364$	8.36
$a_1 + \cdots + a_{2000} = 8.3651$	8.36
$a_1 + \cdots + a_{5000} = 8.36574$	8.36
$a_1 + \cdots + a_{10,000} = 8.36577$	8.36

If a series takes as long to stabilize as this example then, for many purposes, it will not be very practical to use. However, as one adds in more and more turns, **one digit after another has to eventually stabilize.** To repeat: this is because once the k^{th} digit has stabilized, the $k+1^{\text{st}}$ can only stay the same or increase, and it can only increase a maximum of 9 times since there are only 10 possible values for it.

If one is willing to take enough terms, one can find a point at which the first hundred digits of the partial sums have stabilized. Or the first thousand, for that matter.

Now as a practical matter, one is not usually willing to add up an enormous number of terms, and it's often not at all easy to know how many terms one would need to achieve a given degree of accuracy (especially if the partial sums increase steadily but very very slowly).

The point of the above discussion is not to say that it's easy to find the limit of a series by sheer arithmetic. What's at issue is a question of principle, of **theory.**

Namely, we see that

A bounded **positive** series **must converge**
because the sequence of partial sums keeps increasing and so
every decimal place in the sequence of partial sums

will eventually stabilize to some fixed value.

The more digits you want to stabilize, the further you have to go out in the series, and, in most cases, no matter how far out you've gone there will still be digits left that have not yet stabilized.

A Catch. It is not quite correct to say that the decimal places of the partials sums will always eventually stabilize to agree with the decimal places of the limit. For instance if we consider the ruler series, we get

$$\begin{aligned}
&1 = 1 \\
&1 + \frac{1}{2} = 1.5 \\
&1 + \frac{1}{2} + \frac{1}{4} = 1.75 \\
&1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875 \\
&1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375 \\
&1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 1.96875 \\
&1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} = 1.999023438\,.
\end{aligned}$$

The partial sums here will always be slightly less than two, so the units digit of the partial sums will never actually reach 2, and the digits to the right of the decimal place will eventually stabilize at .9999 . . . . The point is, though, that while we cannot say that the decimal expansions of the partial sums eventually reach the true limit, namely 2.0000 . . . , we do see that by taking enough terms the partial sums can be made to agree with the true limit to within any desired degree of accuracy.

The Comparison Test

At first, it seems almost impossible to prove that a series converges without knowing what the limit is. But in fact, there are a number of important **convergence tests** that do just that.

Almost all of the basic convergence tests depend on the principle above: a bounded positive series must converge. Once you realize that what you're really trying to do is to prove that a (positive) series is bounded, the convergence tests start to make a lot more sense.

For instance, there is the **comparison test:**

> If all the terms of a **positive** series $\sum b_n$ are smaller than the terms of a series $\sum a_n$ which is known to converge, then $\sum b_n$ must also converge.

Since for a positive series, converging is the same as being bounded, thus the comparison test can be restated as follows: *If all the terms b_n of a positive series are smaller than the terms of a series $\sum a_n$ which is bounded, then $\sum b_n$ must also be bounded.*

Using the comparison test is often confusing because one is usually trying to compare fractions. It's important to remember the following basic principle.

> Making the denominator of a fraction larger makes the fraction smaller.

Example. The series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \frac{1}{k^2} + \cdots$$

converges.

Proof: Since this is a positive series, it is valid to use the comparison test. Now for $k \geq 2$,

$$\frac{1}{k^2} < \frac{1}{k(k-1)}$$

since $k^2 > k^2 - k = k(k-1)$. But we claim that the series $\sum_2^\infty \frac{1}{k(k-1)}$ converges. This is because of the algebraic identity

$$\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

(For instance, for $k = 3$, $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, and for $k = 8$, $\frac{1}{56} = \frac{1}{7} - \frac{1}{8}$.) When we look at the k^{th} partial sum of the series $\sum \frac{1}{k(k-1)}$ with this identity in mind, we see that

$$\begin{aligned} &\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{k(k-1)} \\ &\quad = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-2} - \frac{1}{k-1}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right). \end{aligned}$$

Now in this sum, each negative terms cancels with the following positive term, so that the entire sum "telescopes" to a value of $1 - \frac{1}{k}$. Since $\lim_{k\to\infty} \frac{1}{k} = 0$, we see that the series converges to 1.

Use of the comparison test now shows that the series $\sum \frac{1}{k^2}$ also converges.

In the same way, one can show that the series $\sum \frac{1}{k^3}$ converges by comparing it with the series

$$\sum_{k=3}^{\infty} \frac{1}{k(k-1)(k-2)} .$$

One can show that this latter series telescopes, although in a more complicated way that the previous example, by using the formula (derived by using partial fractions)

$$\frac{1}{k(k-1)(k-2)} = \frac{\frac{1}{2}}{k-2} - \frac{1}{k-1} + \frac{\frac{1}{2}}{k} .$$

Thus one gets

$$\begin{aligned}
&\frac{1}{6} + \frac{1}{24} + \frac{1}{60} + \frac{1}{120} + \cdots + \frac{1}{k(k-1)(k-2)} + \cdots \\
&\qquad = \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{\mathbf{1}}{\mathbf{10}}\right) + \left(\frac{1}{8} - \frac{\mathbf{1}}{\mathbf{5}} + \frac{1}{12}\right) + \left(\frac{\mathbf{1}}{\mathbf{10}} - \frac{1}{6} + \frac{1}{14}\right) + \cdots \\
&\qquad = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4} ,
\end{aligned}$$

since all the other terms cancel in groups of three. (One group of three canceling terms is indicated in boldface.)

However this approach, while entertaining, is much more work than what is required. The series $\sum \frac{1}{k^3}$ obviously converges by comparison to the series $\frac{1}{k^2}$, since

$$\frac{1}{k^3} < \frac{1}{k^2} .$$

In fact, this logic shows that $\sum \frac{1}{k^p}$ converges whenever $p \geq 2$. (Using the Integral Test — not discussed in these notes — one can show that in fact $\sum \frac{1}{k^p}$ converges if and only if $p > 1$.)

The Limit Comparison Test

The comparison test as it stands is extremely useful and in fact is fundamental in the whole theory of convergence of infinite series. And yet, in a way, it almost misses the point.

In the comparison test, we are comparing the size of the terms in a series of interest with the size of the terms in a series that is known to converge or diverge. However consider the following series:

$$5 + \frac{5}{4} + \frac{5}{9} + \frac{5}{16} + \cdots + \frac{5}{k^2} + \cdots .$$

Now we know that the series $\sum \frac{1}{k^2}$ converges. If we are rather stupid, we might think that this is not very helpful, since $5/k^2$ is not smaller than $1/k^2$. Looking things this way, though, is not using our heads. Obviously $\sum \frac{5}{k^2}$ will converge, and in fact, its limit will be exactly five times the limit of $\sum \frac{1}{k^2}$ (whatever that may be).

What the comparison test in its original form fails to take into consideration is the following important principle:

> What counts is not how big the terms of a series are, but how quickly they get smaller.

Furthermore,

> The convergence or divergence of a series is not affected by what happens in the first twenty or thirty or one hundred or even one thousand terms. The convergence or diverge depends only on the behavior of the **tail** of the series. Therefore if the tail of a series from a certain point on is known to converge or diverge, then the same will be true of the series as a whole.

Taking this into account, we could tweak the comparison test in the following way.

> If $\sum b_n$ is a positive series, and $b_n < Ca_n$ for all n from a certain point on, where C is any positive (non-zero) constant (independent of n) and $\sum a_n$ is a series which is known to converge, then $\sum b_n$ will also converge.
> If, on the other hand, $b_n > Ca_n$ for all n from a certain point on and $\sum a_n$ is known to diverge, then $\sum b_n$ will also diverge.

However one can get an even better tweak than this.

Consider, for example, the series

$$1 + \frac{2}{3} + \frac{3}{11} + \frac{4}{31} + \cdots + \frac{k+1}{k^3+k+1} + \cdots .$$

When k is fairly large (which is what really matters, since we need only look at the tail of the series), $(k+1)/(k^3+k+1)$ is very close to $1/k^2$. Thus it is tempting to compare this series to $\sum \frac{1}{k^2}$, which is known to converge. Working out the inequality is a bit of a nuisance, though, and unfortunately it turns out that $(k+1)/(k^3+k+1)$ is slightly larger than $1/k^2$: just the opposite of what we need.

However, in light of the tweak mentioned above, it would be sufficient to prove that, for instance,

$$\frac{k+1}{k^3+k+1} < \frac{100}{k^2}$$

for large enough k. This is certainly true.

However it seems that one shouldn't have to work this hard. Given that $(k+1)/(k^3+k+1)$ and $1/k^2$ are almost indistinguishable for very large k, and that the tail of the series is all we care about anyway, one would think that if one of the two series $\sum(k+1)/(k^3+k+1)$ and $1/k^2$ converges, then the other should also (although not to the same limit), and if one of them diverges, then they both should.

This is in fact the case. Any time $\lim_{n\to\infty} a_n/b_n = 1$, then two **positive** series $\sum a_n$ and $\sum b_n$ will either both converge or both diverge.

In fact, if we now take into consideration the tweak that we previously made to the limit comparison test (i. e. the observation that what really matters is not how large the terms of a series are, but how fast they get smaller, and that therefore a constant factor in the series will have no effect on its convergence), we get the following:

Limit Comparison Test. Suppose $\sum a_n$ and $\sum b_n$ are **positive** series and that $\lim_{n\to\infty} \dfrac{b_n}{a_n}$ exists (or is ∞).

1. If $\lim_{n\to\infty} \dfrac{b_n}{a_n} < \infty$, and if $\sum a_n$ is known to converge, then $\sum b_n$ also converges.
2. If $\lim_{n\to\infty} \dfrac{b_n}{a_n} > 0$ (or is ∞) and $\sum a_n$ is known to diverge, then $\sum b_n$ also diverges.

Thus the only inconclusive cases are when $\lim_{n\to\infty} b_n/a_n$ does not exist; or when the limit is ∞ and $\sum a_n$ converges; or the limit is 0 and $\sum a_n$ diverges. When the limit is ∞, the terms b_n are so much larger than a_n that $\sum b_n$ might possibly diverge even when $\sum a_n$ converges. And when the limit is 0, the b_n are so much smaller than a_n that $\sum b_n$ might converge even when $\sum a_n$ diverges.

Proof of the Limit Comparison Test. Let's suppose, say, that $\lim_{n\to\infty} b_n/a_n = 5$. This says that if n is large, then b_n/a_n is very close to 5. Then certainly, for large n, $4 < b_n/a_n < 6$. This says that, for large n,

$$b_n < 6a_n \qquad \text{and} \qquad b_n > 4a_n\,.$$

But then if $\sum a_n$ converges, we conclude that $\sum b_n$ also converges, using the tweaked form of the comparison test. And if $\sum a_n$ diverges, then $\sum b_n$ also diverges, for the same reason.

More generally, if $\lim_{n\to\infty} b_n/a_n = \ell > 0$, and we choose positive numbers r and s such that

$$0 < r < \ell < s\,,$$

then for large enough n, b_n/a_n is so close to ℓ that

$$r \ < \ \frac{b_n}{a_n} \ < \ s\,,$$

so that

$$b_n < sa_n \qquad \text{and} \qquad b_n > ra_n$$

for all terms in the series from a certain point on. It then follows from the tweaked form of the comparison test that if $\sum a_n$ converges than so does $\sum b_n$ and if $\sum a_n$ diverges then $\sum b_n$ does as well.

Now consider the possibility that $\lim_{n\to\infty} b_n/a_n = 0$. This would say that for large n, b_n/a_n is very small, so certainly $b_n < a_n$. Thus if $\sum a_n$ converges, then so does $\sum b_n$, according to the comparison test.

And if $\lim_{n\to\infty} b_n/a_n = \infty$, then for large n, $b_n > a_n$, so if $\sum a_n$ diverges then $\sum b_n$ must also diverge.

Mixed Series

If a series has both positive and negative terms, it is called a **mixed series**. The theory for mixed series is more complicated than for positive or negative series, since a mixed series can diverge even though it is bounded both above and below. In this case, we say that it **oscillates**.

A simple example of a series which oscilates is

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots\,.$$

This series is both bounded above and bounded below, since the partial sums never get larger than 1 or smaller than -1. In this case, we find that as we take more and more terms, the partial sums do in fact oscillate between the alternate values $+1$ and 0. As a practical matter, most of the oscillating series one encounters do tend to jump back and forth more or less in this way. However technically, any series which does not go to $+\infty$ or to $-\infty$ and does not converge is called oscillating.

Below, we will distinquish below two different types of convergence for mixed series: absolute convergence and conditional convergence.

The possible behaviors for series are described as follows:

Positive Series	Negative Series	Mixed Series
Converges absolutely	Converges absolutely	Converges absolutely
Goes to $+\infty$	Goes to $-\infty$	Goes to $\pm\infty$
		Oscillates
		Converges conditionally

For a mixed series, we can talk about the **positive part** of the series, consisting of all the positive terms in the series, and the negative part. For instance, in the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

the positive part is

$$1+\frac{1}{3}+\frac{1}{5}+\cdots$$

and the negative part is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots$$

Notice that in writing the negative part, we have taken the absolute value of the terms. Thus we can write

$$\text{Whole Series} = \text{Positive Part} - \text{Negative Part}\,.$$

This is a little misleading, though. It's not always true that in a mixed series the positive terms and negative terms alternate. So when we subtract two series, it's not clear how to interlace the positive and negative terms. For instance, in the example given, we could misinterpret the difference as

$$\text{Positive Part} - \text{Negative Part} = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{4}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\frac{1}{6}+\cdots$$

where there are several positive terms for every negative term. A little thought will show that as long as one keeps including a negative term every so often, all the negative terms will eventually be included in the series so, paradoxically enough, this new series actually contains the same terms as the original one even though the positive terms are being used more rapidly than the negative ones.

One's first impulse is to think that changing the way the positive terms and negative terms of a series are interlaced shouldn't make any different to the limit, since both series ultimately do contain the same terms. However consideration of the partial sums seems to clearly indicate that the two series do not have the same limit. In fact, the second series

$$\text{Positive Part} - \text{Negative Part} = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{4}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\frac{1}{6}+\cdots$$

does not seem to converge at all, whereas we shall see from the Alternating Series Test below that the first one does.

This is the reason that one should not think of an infinite series as merely a process of adding up an infinite number of terms. Instead, it is a process of adding more and more terms taken in a particular sequence.)

Here are the possibilities for a series with both positive and negative terms.

1. The positive part of the series and the negative part both converge. In this case, the series as a whole must converge.
2. The positive part converges, but the negative part diverges. In this case, the series as a whole must diverge. More precisely, as one adds on more and more terms, the result becomes more and more negative, i. e. the sum "goes to $-\infty$."
3. The positive part diverges and the negative part converges. Once again, the series as a whole diverges. In this case, it goes to $+\infty$.
4. The positive part and the negative part both diverge. **In this case, anything can happen.**

	Positive part Converges	Positive Part Diverges
Negative part Converges	Series Converges Absolutely	Series Diverges
Negative part diverges	Series Diverges	Series diverges or Converges conditionally

At first, it seems very unlikely that a series can converge if its positive and negative parts are both diverging. What happens, though, is that as one adds more and more terms, even though the positive terms alone add up to something which eventually becomes huge, and the negative terms add up to something which becomes hugely negative, as one goes down the series the two sets of terms keep balancing each other out so that one gets a finite limit.

Alternating Series

In practice, mixed series are not usually as troublesome as the discussion above would suggest. This is because in most mixed series, the positive and negative terms alternate. In this case, what usually happens is that either the series obvious diverges (in fact, oscillates) because $\lim_{n\to\infty} a_n \neq 0$, or else it converges (either absolutely or conditionally) according to a very simple test, which will be described below.

Consider the following series, whose positive and negative parts both diverge.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

SIDEBAR: Decimal Representations As Infinite Series

We usually take if for granted that a real number is given in the form of a decimal. But this leaves the problem of explaining just exactly what we mean by a decimal number which has infinitely many decimal places. We can explain 3.48, for instance, as a shorthand for $\dfrac{348}{100}$. But what is $1.2345678910111213\dots$ a shorthand for?

It seems clear that a workable explanation can only be given in terms of the limit concept. The idea of an infinite series is one way of giving such an explanation.

For instance, the decimal expanion for π,

$$\pi = 3.14159265\dots$$

can be interpreted as the infinite series

$$3 + x + 4x^2 + x^3 + 5x^4 + 9x^5 + 2x^6 + 6x^7 + 5x^8 + \cdots$$

where $x = 1/10$.

This is particularly useful in the case of decimals such as

$$.001001001001\dots.$$

We can interpret this as

$$\frac{1}{1000}(1 + x + x^2 + x^3 + x^4 + \cdots),$$

with $x = \frac{1}{1000}$. Since the expression in parentheses is a geometric series, we can evaluate .001001. . . . as

$$.001001001\cdots = \frac{1}{1000}\,\frac{1}{1-x} = \frac{1}{1000}\,\frac{1}{.999} = \frac{1}{999}\,.$$

From this, we can see that any repeating decimal with the pattern $.xyzxyz\dots$ evaluates to $xyz/999$. For instance,

$$.027027027\cdots = 027 \times .001001001 = \frac{027}{999} = \frac{1}{37}\,.$$

It's great that infinite series give us a way of actually explaining what a non-terminating decimal really means. On the other hand, there is a certain amount of circular reasoning here. We explain what it means for an infinite series to converge by saying that it converges to a real number. And then we explain what a real number is by thinking of it in terms of its decimal expansion. And now we explain what a decimal expansion is by interpreting it as an infinite series. This is enlightening, and sometimes useful, but hardly adequate for a rigorous foundation of mathematical analysis.

Also note that if wee take the method described above for evaluating repeating decimals, and apply it to $.999999999\dots$, we get the apparently paradoxical (but true)

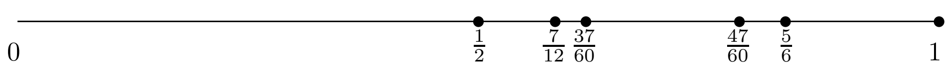
$$.9999999999\cdots = \frac{999}{999} = 1\,,$$

so that in cases like this, two different decimal expansions can correspond to the same real number.

If one looks at what happens as one adds in more and more terms of this series, one gets the following partial sums:

$$
\begin{aligned}
&1 \\
1-\frac{1}{2} &= \frac{1}{2} = \frac{30}{60} \\
1-\frac{1}{2}+\frac{1}{3} &= \frac{5}{6} = \frac{50}{60} \\
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} &= \frac{7}{12} = \frac{35}{60} \\
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} &= \frac{47}{60} \\
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6} &= \frac{37}{60}
\end{aligned}
$$

$$\cdots$$



Each new term being added on has the opposite sign of the one before, so it takes the sum in the opposite direction to the previous one: if the previous term was positive and moved the sum towards the right, then the new term will be negative and move it towards the left. As the fractions get more complicated here, it becomes difficult to visualize their relative positions on the number line, but it's not hard to show that what happens is that instead of getting large and larger, the partial sums are jumping back and forth within a smaller and smaller radius. On the other hand, the jump to the left will be smaller than the previous jump to the right, because the terms a_k keep getting smaller (in absolutely value). Since the partial sums of this series keep jumping back and forth within a space whose radius is converging to 0, one's intuition suggests that the series must eventually converge to some limit.

Now anyone who goes through a calculus sequence paying careful attention to the theory will eventually realize the general principle that intuition is very often wrong. However in this case we have an exception. Intuition is correct, and any series of this kind does converge.

A series which changes sign with each term — i.e. the sign of each term is the opposite of the sign of the preceding one — is an **alternating series**. Not every mixed series is an alternating series, but a lot of the most important ones are.

Alternating series are particularly nice because of the following:

> **Alternating Series Test:** An alternating series will always converge any time both the following two conditions hold:
>
> (1) Each term is smaller in absolute value than the one preceding it;
>
> (2) As k goes to ∞, the k^{th} term converges to 0.

Neither one of these two conditions is adequate without the other. For instance, the series

$$1.1 - 1.01 + 1.001 - 1.0001 + 1.00001 - 1.000001 + \cdots$$

is alternating and clearly does not converge, even though each term is smaller in absolutely value than the preceding one.

On the other hand, the alternating series

$$1 - 1 + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{25} + \frac{1}{5} - \frac{1}{5^3} + \frac{1}{6} - \cdot$$

fails the alternating series test because

$$\begin{array}{c} \frac{1}{3} \not\leq \frac{1}{5} \\ \frac{1}{6} \not\leq \frac{1}{5^3} \\ \text{etc.} \end{array}$$

even though the n^{th} term does go to zero as n increases. A series like this might still converge, but this particular one does not. (The negative part converges and the positive part diverges, so the series as a whole must diverge.)

An annoying thing about using infinite series for practical calculuations is that even though you know that by taking enough terms of a convergent series you can get as close to the limit as you want, in many cases it's not very easy to figure out just exactly how many terms you'll need to achieve some desired degree of accuracy.

But one of the nice things about series for which the alternating series test applies is that, since the partial sums keep hopping back and forth from one side of the limit to the other in smaller and smaller hops, you can be sure that the error at any given stage is always less than the size of the next hop: i. e. less than the absolute value of the next term in the series.

Let

$$a_1 - a_2 + a_3 - a_4 + a_4 + \cdots$$

be an alternating series satisfying the two conditions of the alternating series test. Then for any n, the difference between the partial sum

$$a_1 - a_2 + \cdots \pm a_n$$

and the true limit of the series is always smaller than $|a_{n+1}|$.

In fact, for many alternating series that one actually works with in practice, once one goes a way out in the series, the size of each term is not much different from the size of the preceding one. This means that as partial sums hop back and forth across the limit, the forward hops and the backward ones are roughly the same size. This suggests that the true limit should lie roughly halfway between any two successive partial sums. In other words, the error obtained by approximating the series by the n^{th} partial sum will be roughly $|a_{n+1}|/2$. (However one can easily cook up contrived examples where this is not a good estimate.)

The frustrating thing here, though, is that for some of the most well known alternating series, this criterion shows that the error after a reasonable number of steps is still discouragingly large. For instance, the following alternating series (derived from the Taylor series for the arctangent function) converges to $\pi/4$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$

One might hope that we could get a pretty accurate approximation by taking 100 terms of this series. But the 100^{th} term here will be $-1/101$, and so the theorem above only guarantees us that after taking 100 terms, the error will be smaller than $1/103 \approx .01$. In other words, the theorem only tells us that after taking 100 terms of the series, we can only be sure of having an accurate result up to the second digit after the decimal point. If we want to be sure of accuracy up to the sixth digit after the decimal point, the theorem says that we would need to take a million terms of the series.

Now mathematicians are not bothered by the idea that one needs to take a million terms of a series to get reasonable accuracy — they're not going to actually **do** the calculation, they're just going to **talk** about it. For end users of mathematics, though — physicists, engineers, and others who walk around carrying calculators — this sort of accuracy (or rather lack thereof) is anything but thrilling. These people prefer to work with series where one gets accuracy to at least a couple of decimal places by taking the first two or three terms, not half a million.

Of course if we use the idea that the actual error for the alternating series that one usually encounters is likely to be roughly half the next term in the series, this would suggest that to get accuracy to the sixth place after the decimal point in the above series for $\pi/4$, one should really only need a *half* of a million terms. What a thrill!

However if it's really true that the limit for the most typical alternating series is often about halfway between two successive terms, then we ought to be able to get much better accuracy by replacing the final term in the partial sum by half that amount. In other words, we could try a partial sum of

$$a_1 - a_2 + a_3 - \cdots \pm a_{n-1} \mp \frac{a_n}{2}\,.$$

Suppose we try this with the series for $\pi/4$, this time taking only ten terms. A calculation shows that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{2}\cdot\frac{1}{19} \approx .7868\,.$$

On the other hand, to four decimal places, $\pi/4 = .7854$. So in this example, at least, by tweaking the calculation we got accuracy up to an error of roughly .001 using only 10 terms of the series, instead of needing five hundred thousand.

Tweaking an alternating series in this way is likely to often give fairly good results when a_{n+1} and a_n are roughly the same size (at least for large n), although without a theorem to justify one's method, one doesn't have guaranteed reliability.

On the other hand, consider the alternating series

$$\sum_0^\infty \left(\frac{-1}{5}\right)^n = 1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \frac{1}{625} - \cdots + (-1)^n\frac{1}{5^n} + \cdots\,.$$

Look at some partial sums for this series:

$$\begin{aligned}
1 &= 1\\
1 - \frac{1}{5} &= 1 - .2 = .8\\
1 - \frac{1}{5} + \frac{1}{25} &= .8 + .04 = .84\\
1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} &= .84 - .008 = .832\\
1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \frac{1}{625} &= .832 + .0016 = .8336\,.
\end{aligned}$$

Here a_{n+1} is much smaller than a_n (in fact, $a_{n+1} = a_n/5$). This is a geometric series and its limit is $\dfrac{1}{1+\frac{1}{5}} = \dfrac{5}{6} = .833333\ldots$. If we were to use $a_0 + a_1 + a_2 + a_3 + \dfrac{a_4}{2}$ as an approximation to the limit we would wind up with a value of .8328, which is not nearly as good an approximation as $a_0 + a_1 + a_2 + a_3 + a_4 = .8336$.

Obviously no series for which $\lim_{k\to\infty} a_k \neq 0$ can ever converge. On the other hand, occasionally one will encounter an alternating series where the successive terms do not consistently get smaller in absolute value. If a series like this does not converge absolutely, it may be quite a problem figuring out what happens.

Absolute Convergence

For mixed series, we distinguish between two types of convergence: **conditional** convergence and **absolute** convergence. The issue here is whether the terms of the series get small so rapidly that it would converge even if we ignored the signs, or if the terms of the series get small slowly but the series still converges only because the positive and negative terms remain in balance.

Let's restate this more carefully. If we take a mixed series and make all the terms positive, then we get the corresponding **absolute value series**. The absolute value series is obtained from the original series by adding together the positive and negative parts instead of subtracting them.

$$\begin{aligned} \text{Original Series} &= \text{Positive Part} - \text{Negative Part} \\ \text{Absolute Value Series} &= \text{Positive Part} + \text{Negative Part} \end{aligned}$$

Absolute convergence means that the absolute value series converges. (Conditional convergence will be defined below as meaning that the original mixed series converges, but the corresponding absolute value series does not.)

(For the record, we note the following trivial fact: **Any positive series converges if and only if it converges absolutely. Likewise for a negative series.**)

The definition does not explicitly say that a series which converges absolutely actually does converge, however this is in fact the case. To see this, we can note the following important principle.

> If a positive series converges, and a new series is formed by leaving out some of the terms of this series, then the new series will also converge.

The reason for this is that saying that a positive series converges is the same as saying it is bounded. But leaving out some of the terms of a bounded series can't possibly make it become unbounded.

Since the absolute value series corresponding to an original mixed series is the sum of the positive and negative parts of the original series, the above principle shows that the absolute value series corresponding to a given series converges if and only if the positive and negative parts of the series both converge. From this, we see the following:

> If a series converges absolutely, then it converges.

The limit comparison test can sometimes be used to determine whether an infinite series converges absolutely or not.

New Limit Comparison Test Suppose $\sum a_n$ and $\sum b_n$ are not-necessarily-positive series and that $\lim_{n\to\infty} \left|\frac{b_n}{a_n}\right|$ exists (or is $+\infty$).

1. If $\lim_{n\to\infty} \left|\frac{b_n}{a_n}\right| < \infty$, and if $\sum a_n$ is known to converge absolutely, then $\sum b_n$ also converges absolutely.
2. If $\lim_{n\to\infty} \left|\frac{b_n}{a_n}\right| > 0$ (or is ∞) and $\sum a_n$ is known to not converge absolutely, then $\sum b_n$ also does not converge absolutely.

As indicated above, a series can sometimes converge even when its positive and negative parts both diverge i. e. without converging absolutely. This can happen because as we go further and further out in the series, the positive and negative terms balance each other out.

For instance the alternating series discussed above,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges but does not converge absolutely, since the absolute value series is the divergent Harmonic Series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

In a case like this, although the convergence is quite genuine, it is also rather delicate, since it depends on the positive and negative terms staying in balance. If we were to rearrange the order of the series, for instance,

$$\text{Positive Part} - \text{Negative Part} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{4} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} - \frac{1}{6} + \cdots$$

the new series would not converge, since the positive terms would outweigh the negative ones. And yet both series consist of the same terms, only arranged in a different order. (At first, one is likely to think that some negative terms will get left out of the second series, since the positive terms are being "used up" much more quickly than the negative ones. But in fact, every term in the original series, whether positive or negative, does eventually show up in the new one, although the negative ones show up quite a bit further out than originally.)

When a series converges, but the corresponding absolute value series does not converge, one says that the series **converges conditionally**.

This term is unfortunate, because it leads students to think that a series which converges conditionally doesn't "really" converge. It does quite genuinely converge, though, as long as one takes the terms in the order specified. Rearranging the order of terms, however, will cause problems in the case of a conditionally converging series. The rearranged series may diverge, or it may converge to some different limit, since rearranging changes the balance between positive and negative terms.

This is a key reason to remember than when one finds the limit of an infinite series one is not actually adding up all the infinite number of terms. In evaluating an infinite series, one is doing calculus, not algebra. In algebra, one always gets the same answer when adding a bunch of numbers, no matter what order one adds them in. In infinite series, the order of the terms can effect what the limit is, if the series converges conditionally.

In fact, there's a theorem due to Riemann that says that by rearranging the terms of a series which converges conditionally, you can make the limit come out to anything you want.

Theorem [Riemann]. By rearranging the terms of a conditionally convergent series, you can get a series that diverges or converges to any preassigned limit.

PROOF: The point is that if a series converges conditionally, then the positive and negative parts both diverge, but they are in such a careful balance that the difference between them converges to some finite limit. Rearranging the series will affect this balance, and with sufficient premeditation one can make the limit come out to anything one wants.

Let's consider, for instance, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

which is known to converge to $\ln 2$. The positive part of this series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

and the negative part

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$$

both diverge. This is an essential requirement for the trick we shall use to work. It is also essential to know that the limit of the n^{th} term as n goes to infinity is 0. This would always be the case, otherwise the series could not converge even conditionally.

Now suppose we want to rearrange this series to get a limit of, say, -20. Considering the size of the terms we're working with, -20 is a really hugely negative number, but we could even go for -500, if we really wanted.

To start with, we'll use only negative terms of the series. Since the negative part of the series diverges, we know that by taking enough terms

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \cdots$$

we can eventually get a sum more negative than -20. It would be rather painful to calculate exactly how many terms we'd need to get this, and it really doesn't matter, but just out of curiosity we can make a rough approximation. Our guesstimate for the required number of terms will be based on the fact that the sum of the third and fourth terms here is numerically larger than $1/4$ (i. e. larger in absolute value), since $1/6$ is larger than $1/8$. Likewise the sum of the next four terms is numerically larger than $1/4$, since

$$\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4}.$$

Continuing in this way, the sum of the next eight terms after that is numerically larger than $1/4$, as is the sum of the next sixteen terms after that.

Now $-1/4$ is not a very negative number in comparison to -20, but by the time we get 80 groups of terms, all less than $-1/4$, we'll have a sum of less than -20. A careful consideration thus shows that

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{2^{80}} < -20.$$

Note that $2^{80}=\left(2^{10}\right)^8$, and $2^{10}=1024$, so that 2^{80} is not a whole lot bigger than $(10^3)^8=10^{24}$. Thus it looks like it will take roughly an octillion negative terms to push the partial sum below -20. But nobody promised it was going to be easy! After all, we're trying to produce a large number (or rather an extremely negative one) by adding up an incredible number of small ones.

At this point, we can now finally include a positive term, so that the rearranged series so far looks something like

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots-\frac{1}{2^{80}}+1.$$

Now this positive term will undoubtedly push the sum back up above -20. If not, we can include still more positive terms until we achieve that result. The crucial thing is that no matter how big a push we need, there are enough positive terms to achieve that, since we know the positive part of the series diverges.

Once we manage to push the sum above -20, we start using negative terms again to push it back down. And once the sum is less than -20, we use positive terms again to push it back up to something greater than -20.

As we keep making the partial sums keep swinging back and forth from one side of -20 to the other, it's essential to make sure that the radius of the swings approaches 0 as a limit, so that the new series actually converges to -20. We can accomplish this by always changing direction (i. e. changing from negative to positive or vice-versa) as soon as the partial sum crosses -20, since in this case the difference between the partial sum and -20 will always be less than the absolute value of the last term used, and we know that the last term will approach 0 as we go far enough out in the series.

Now one can't help but notice that in the above process one is using up the negative terms at an extravagantly lavish rate, and using positive terms at an extremely miserly rate. So one's first thought is that either one eventually runs out of negative terms, or that sum of the positive terms never get included at all.

In the first place, though, one never runs out of negative terms because there are an infinite number of them available.

But do all of the positive terms eventually get used? Well, choose a positive term and I'll convince you that it does eventually appear in the new series. Suppose, say, you choose $\frac{1}{201}$. This is the 100^{th} positive term in the original series. Now if we never got to $\frac{1}{201}$ in the rearranged series, this would mean that at most 99 positive terms from the original series are being used for the new series. But this means that in the new series, the positive terms would add up to less than

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{201}\,.$$

Now we don't need to worry about just exactly how big that sum is, because the point is that it's some finite number, even if possibly somewhat large. On the other hand, the negative part of the series diverges. This would mean that our new series would eventually start becoming indefinitely negative, approaching $-\infty$, which would violate our game plan, because once the partial sum is less than -20, we are supposed to use another positive term. So the point is, if we follow the game plan then we can be sure that $\frac{1}{201}$, or in fact any of the positive terms, does eventually occur.

Using variation on this method, we can produce a rearranged series that diverges. We start out as before, using negative terms until the partial sum is less than -20. Then we use one positive term, then use more negative terms until the sum is less than -40. Again we use one positive term, then go back to negative ones until the sum is less than -60. Etc. etc. Once again, one can see that even though one seems to almost never use any positive terms, eventually all the positive terms of the original series do get included in the rearranged series.

Hilbert's Infinite Hotel. Riemann's trick, as described above, depends on properties of infinite sets that mathematicians were only beginning to appreciate in the Nineteenth Century.

Around the beginning of the Twentieth Century, Hilbert explained this basic idea as follows: Suppose that we have a hotel with an infinite number of rooms, number $1, 2, 3, \cdots$, and all the rooms are full. (This is a purely imaginary hotel, of course, because infinity does not occur in the real world. If modern physics is correct, even the number of atoms in the whole universe, although humongous, is still not infinite.)

Now suppose a new guest shows up. In the real world, since all the rooms are full, there would be no room for the new guest. But in the infinite hotel, one simply has the guest in room 1 move into room 2, and the guest in room 2 move into room 3, and the guest in room 4 move into room 5, etc. Everytime one has the guest in room n move, there is always a room $n+1$ to move him into.

(One might even be able to get away with this in a real world hotel if there were enough rooms. Say there were a thousand rooms. Before one got to room 1000, where there would be a problem, surely someone would have checked out.)

Hilbert's Infinite Hotel is a little like a Ponzi scheme, where one sells a worthless security to investors, but keeps paying off the old investors by using the money paid in by the new ones. Ponzi schemes don't work because the real world is not infinite, so eventually one runs out of new suckers, er, investors to supply the necessary money.

One of the miracles of modern mathematics is that it manages to use something that can't exist in the real world — infinity — to achieve results that do work in the real world.

Absolutely Convergent Series (continued). To see that what happens in conditionally convergent series cannot happen for absolutely convergent ones, let's first consider the case of a positive series. In the case of a series whose terms are all positive, one cannot affect whether the series converges or not by rearranging the terms. This is because a positive series converges if and only if it is bounded, and you can't change whether a series is bounded or not by taking the same terms in a different order. Not only that, but you can't affect what the limit is by rearranging the terms of a positive series. To see why this is, let's consider the example of the Ruler Series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

The limit of this series is 2 and by the time one has taken the first 8 terms, the sum agrees with the limit to within an error smaller than .01:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 1\frac{127}{128}\,.$$

Now this means that all the terms from the 9^{th} one out never add up to more than .01 (in fact, never add up to more than 1/128), no matter how many one takes. Now suppose we take the same terms in some different order, but without repeating any or leaving any out. Eventually, if we go far enough out in the rearranged series we will have to include all the first 8 terms $1,\ 1/2,\ 1/4,\ 1/8,\ \ldots,\ 1/128$ of the original series (and probably many others as well), because by assumption no terms are getting left out. For instance, the beginning of the rearranged series might look like

$$\frac{1}{4} + \frac{1}{512} + \frac{1}{32} + \frac{1}{2} + 1 + \frac{1}{128} + \frac{1}{64} + \frac{1}{1024} + \frac{1}{16} + \frac{1}{8}\,.$$

Now note that

$$2 \quad > \quad \frac{1}{4} + \frac{1}{512} + \frac{1}{32} + \frac{1}{2} + 1 + \frac{1}{128} + \frac{1}{64} + \frac{1}{1024} + \frac{1}{16} + \frac{1}{8} \quad > \quad 1\,\tfrac{127}{128}\,.$$

(The first inequality is true because the limit of the original series is 2 and the sum in the middle does not have all the terms of the complete series. The second inequality is true because the sum in the middle is larger than the sum of the terms from 1 to 1/128.) But at that point, whatever terms are left will add up to less than .01. That means that eventually the sum of the rearranged terms will be within .01 of the original limit 2. So the new limit and the original limit must agree to with a possible error of .01.

But .01 was nothing except an arbitrarily convenient standard of accuracy. By going far enough out in the series, we could replace this by an desired small number. Thus we can show that the limit of the rearranged series and the limit of the original series agree to within any conceivable degree of accuracy. Thus they are the same.

The logic here applies to any series whose terms are all positive.

> Rearranging the terms of a positive series does not affect whether the series converges or not, and if it does converge, rearranging the terms does not affect what the limit is.

Now, going back to mixed series, what we see is that if we rearrange a series, this will not affect the limit of the corresponding absolute value series. Therefore if the mixed series in question converges absolutely, no matter how one rearranges the terms, the new series would still converge absolutely and thus could not diverge. And furthermore, even when an absolutely convergent series is rearranged, if one goes far enough out in the series then all the positive terms which are still left as well as all the negative terms still left will only add up to something extremely small. In fact, the same of the terms at the end can be made arbitrarily small by going far enough out in the series. Thus essentially the same logic as given above for positive series applies to show that

> Rearranging the terms of an absolutely convergent series does not affect whether the series converges or not, and if it does converge, rearranging terms does not affect what the limit is.

The Ratio Test

Consider the series

$$72 + 36 + 12 + 3 + \frac{3}{5} + \frac{1}{10} + \frac{1}{70} + \frac{1}{560} + \cdots$$

The pattern here is that the second term is one-half the first term, the third term is one-third the second term, the fourth term is one-fourth the third term, and for all k, $a_k = a_{k-1}/k$. The numbers in this series are fairly large, which makes it seem unlikely that the series converges. However we can notice that since $\frac{1}{k} < \frac{1}{2}$ for $k > 2$, each term of this series after the second is smaller than one-half the preceding term, so that

$$72 + 36 + 12 + 3 + \frac{3}{5} + \frac{1}{10} + \frac{1}{70} + \cdots < 72\,(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \cdots) = 72 \times 2,$$

showing that this series converges by the comparison test with respect to the geometric series for $x = 1/2$. (The comparison test is valid since the series is positive.)

In general, suppose that we have a **positive** series

$$a_0 + a_1 + a_2 + a_3 + \dots$$

such that for all k, $a_{k+1}/a_k < r$, where r is a positive number strictly smaller than 1. Then $a_{k+1} < ra_k$ and so

$$a_0 + a_1 + a_2 + \cdots < a_0(1 + r + r^2 + r^3 + \cdots)$$

and therefore the series converges by comparison with the geometric series.

Conversely, if $a_{k+1}/a_k > r$ for some positive number r with $r \geq 1$, then clearly the series cannot converge. (In this case, the terms aren't even getting smaller.)

This is the crude form of the **ratio test.**

There are two pitfalls to the crude form of the ratio test. First, the reasoning here does not justify applying it to series which are not positive, since the comparison test only works for positive series. (We will later see a way around this restriction.) Secondly, it is not enough to merely know that $a_k/a_{k+1} < 1$ for all k. One must have a positive number r **strictly smaller** than 1 which is independent of k such that $a_{k+1}/a_k < r$ for all k.

As an example to illustrate this second pitfall, consider the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots .$$

For this positive series, $a_{k+1}/a_k < 1$ for all k, yet the series diverges.

There are two ways of tweaking the ratio test. First, since the convergence or divergence of a series won't be affected by what happens in the first few terms (or first 100 terms, or even first 1000 terms),

> instead of requiring that $a_{k+1}/a_k < r$ for all k, it is sufficient to know that this is true *from a certain point on*, i. e. for all *sufficiently large* k.

Thus if we know that $a_{k+1}/a_k < .9$ for all k larger than 1000, then the series converges.

Consider, for instance, the series

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{k!} + \cdots .$$

For this series, we have $a_{k+1} = a_k/(k+1)$, so that surely $a_{k+1} < \frac{1}{2}a_k$ for $k = 3, 4, \cdots$. Therefore the series converges. (In fact, it is known to converge to $e = 2.718\dots$.)

Now consider the series

$$1 + 100 + \frac{100^2}{2} + \frac{100^3}{6} + \frac{100^4}{24} + \cdots + \frac{100^k}{k!} + \cdots$$

The first few terms of this series (in fact the first hundred or so) are completely humongous, so it seems to have no change of converging. And yet for $k > 200$ we have $a_{k+1} = 100a_k/k < \frac{1}{2}a_k$, so that the ratio test shows that the series does in fact converge. (In fact, it converges fairly rapidly, once one gets past the first hundred terms or so, which are indeed huge.)

The second tweak is even more powerful, despite being essentially a special case of the first.

> One can actually take the limit of a_{k+1}/a_k as k gets larger and larger. If this limit is **strictly** smaller than 1, then the series converges.

To see why this is so, suppose, for instance, that $\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = .9$. This means that by taking k large enough, we can make a_{k+1}/a_k arbirarily close to .9. For instance, from some point on, all the values of a_{k+1}/a_k will lie within a distance smaller than .05 from .9. Restated, this says that

$$.85 < \frac{a_k}{a_{k+1}} < .95$$

from some point in the series on. But this means that the series converges by the ratio test with $r = .95$.

In more general terms, the reasoning is that if $\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \ell$ and $\ell < 1$, then there exist a real numbers r between ℓ and 1. Furthermore, if we write $\varepsilon = r - \ell$, then $\varepsilon > 0$ and by definition of the concept of limit, whenever k is large enough then

$$\ell - \varepsilon < \frac{a_{k+1}}{a_k} < \ell + \varepsilon = r\,.$$

But then since $r < 1$, the series converges by the crude form of the ratio test.

The flip side of the ratio test also works. Namely, if $\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} > 1$, then the series diverges. This is because if $\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \ell > 1$, and if s is any number such that $\ell > s > 1$, then applying the same kind of reasoning as above it can be seen that $a_{k+1}/a_k > s$ for all k from a certain point on. But since $s > 1$, this says that for all k from a certain point on, $a_{k+1} > a_k$, so surely the series cannot converge. (Note that this reasoning can be applied even if the series is not positive, provided we look at $\lim_{k\to\infty} |a_{k+1}/a_k|$.)

It turns out the the ratio test works even for series which are not positive, if we consider $\lim_{k\to\infty} |a_{k+1}/a_k|$. If this limit is strictly smaller than 1, this will in fact show that the series $|a_0| + |a_1| + |a_2| + |a_3| + |a_4| + \cdots$ converges, so that the original series converges absolutely, and thus converges. On the other hand, if $\lim_{k\to\infty} |a_{k+1}/a_k| > 1$, this means that $|a_{k+1}| > |a_k|$ for all large k, so certainly the original series can't converge.

> **Ratio Test.** If $a_0 + a_1 + a_2 + \cdots$ is an infinite series, let $\ell = \lim_{k\to\infty} \dfrac{|a_{k+1}|}{|a_k|}$. If $\ell < 1$ then the series converges and if $\ell > 1$ then the series diverges.

The ratio test is a marvelous trick because it enables one to decide whether a vast number of series converge or not without doing any hard thinking, provided that one can compute ℓ, which is often not very difficult. (In fact, it's so powerful that maybe it should be outlawed for students. ☺)

The only drawback to the ratio test is that it doesn't give any information in case $\ell = 1$. In this case, the decision could go either way. Consider, for instance, the following three series:

$$1+2+3+4+\cdots$$
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$
$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{k^2}+\cdots$$

The first series obviously diverges, and the second is the Harmonic Series, which also diverges. The third series is known to converge. And yet for all three of these series, $\ell = \lim_{k\to\infty} a_{k+1}/a_k = 1$.

POWER SERIES

The idea of a **power series** is a variation on the geometric series. Instead of just considering the geometric series

$$1+x+x^2+x^3+x^4+\cdots,$$

one can allow the powers of x to have coefficients:

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots.$$

As a simple example, suppose we set $a_n = 3^n$ for all n. Then

$$\sum_0^\infty a_nx^n = \sum_0^\infty 3^nx^n = \sum_0^\infty (3x)^n.$$

This is just the Geometric Series for the variable $3x$, hence it converges for to $1/(1-3x)$ for $|3x| < 1$, i. e. for $-\frac{1}{3} < x < \frac{1}{3}$, and diverges for $|x| \geq \frac{1}{3}$.

Power series can be much more complicated than this. And yet this sort of Geometric Series is in some sense a model for the convergence of every power series.

It seems clear that whether a power series

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$$

converges or not for a certain value of x will depend both on the size of x and on the size of the coefficients a_n or, more precisely, on how fast they get small or how slowly they get large. For instance in the series

$$1+3x+9x^2+\cdots+3^nx^n+\cdots,$$

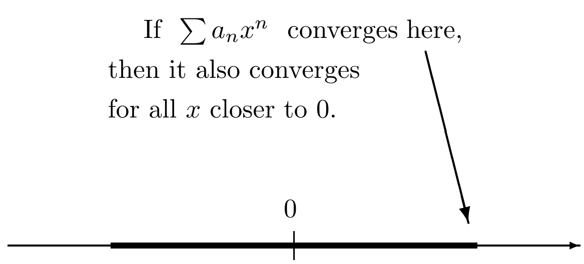
the coefficients get large fairly quickly, so we would expect that x would have to be fairly small for the series to converge. And, in fact, we know that the series converges only for $-\frac{1}{3} < x < \frac{1}{3}$. On the other hand, in the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots ,$$

we have $a_n = a_{n-1}/n$, so the coefficients a_n eventually get small quite rapidly. (And "eventually" is what matters!) Thus we might expect that the series could converge even when x is comparatively large. In fact, this series is known to converge for all x.

It seems extremely plausible that if a power series $\sum a_n x^n$ converges for a certain value of x, then it would also converge for smaller values, since in this case the terms $a_n x^n$ would be smaller (in absolute value).

If this is in fact the case, then a consequence would be that the set of values of x where $\sum a_n x^n$ converges would form an interval around the origin.



If, furthermore, convergence of $\sum a_n x^n$ depends only on $|x|$, then this interval of convergence would be symmetric around 0.

The catch in this reasoning is that the convergence of a series does not always depend solely on the size of the terms (in absolute value). In the case of conditional convergence, the balance between positive and negative terms is also a crucial factor, and it is not completely clear how changing the size of x will affect the balance between positive and negative terms in $\sum a_n x^n$.

And in fact, the set of points x where $\sum a_n x^n$ converges is not always absolutely symmetrical around 0. For instance, $\sum \dfrac{(-1)^n x^n}{n}$ converges when $x = 1$ by the alternating series test, but when $x = -1$ it becomes the the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots ,$$

and therefore diverges.

Fortunately, however, it turns out that power series either converge absolutely whenever they converge at all, or converge conditionally at only one or two point: one or both of the endpoints of the interval of convergence for the series.

Thus, for instance, from the very fact that $\sum \frac{(-1)^n x^n}{n}$ converges conditionally when $x = 1$, we will be able to immediately conclude that the interal of convergence for this series consists of those x with $-1 < x \leq 1$.

We now proceed to the proof of this assertion, as well as the proof that the set of points x where a power series converges does in fact constitute an interval which is symmetric around 0 except possibly for the endpoints.

For a fixed x, consider the following five possibilities for the series $\sum a_n x^n$, arranged in order of increasing desirability.

1. $\lim_{n\to\infty} |a_n x^n| = \infty$ (or, more generally, the set of numbers $|a_n x^n|$ is not bounded), so clearly the series does not converge.
2. $\lim_{n\to\infty} |a_n x^n| < \infty$ (or, more generally, the set of numbers $|a_n x_n|$ is bounded) but $\lim_{n\to\infty} |a_n x^n| \neq 0$, so the series can't possibly converge.
3. $\lim_{n\to\infty} |a_n x^n| = 0$, but the series still doesn't converge.
4. The series converges conditionally.
5. The series converges absolutely.

Possibilities number (1) and (5) are the most clear-cut cases. Possibilities number (2), (3) and (4) are the most difficult to recognize. How fortunate it is, then, as we shall see, that possibilities (1) and (5) are the ones most commonly encountered, and possibilities (2), (3) and (4) can occur for only rare values of x.

Most specifically, if $\lim_{n\to\infty} |a_n x_0^n| < \infty$ but $\sum a_n x_0^n$ diverges or if $\sum a_n x_0^n$ converges conditionally, then the series $\sum a_n x^n$ **converges absolutely** for all x with $|x| < |x_0|$ and **diverges** for all x with $|x| > |x_0|$.

> The set of x where a power series $\sum a_n x^n$ converges is either the whole real line $\mathbb{R}$, or consists of the single point $\{0\}$, or is an interval centered around 0, whether open, closed, or half-open. (All these possibilities do occur.)
>
> If $\sum a_n x_0^n$ converges conditionally or $\lim_{n\to\infty} |a_n x_0^n| < \infty$ but $\sum a_n x_0^n$ diverges, then x_0 is one of the two endpoints bounding the interval where $\sum a_n x^n$ converges.

In some cases, this fact enables us to instantly see for what values of x a series converges and for what values it diverges. Consider the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots .$$

For $x = 1$ this converges by the alternating series test. But it does not converge absolutely, since the corresponding absolute value series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. Therefore, the principle we have claimed above this enables us to conclude that the series converges for all x with $|x| < 1$ and diverges for all x with $|x| > 1$. As we have seen, the series converges for $x = +1$, and it diverges for $x = -1$ because in this case it becomes the negative of the Harmonic Series.

It remains to see why this extremely useful principle is true. It is in fact a consequence of an even more far-reaching principle.

Proposition [Abel]. Suppose that there is an upper bound B such that $|a_n x_0^n| < B$ for all n. (This is essentially equivalent to the statement that $\lim_{n\to\infty} |a_n x_0^n| < \infty$, except in cases where the limit does not exist.) Then there exist positive numbers r and B such that

$$\left|\frac{x_0}{r}\right| \le 1 \qquad \text{and} \qquad |a_n| \le \frac{B}{r^n} \quad \text{for all } n.$$

Furthermore the series $\sum a_n x^n$ may or may not converge when $x = x_0$, but it **converges absolutely** whenever $|x| < |x_0|$.

In particular, this is the case if $\sum a_n x_0^n$ converges, whether absolutely or conditionally.

PROOF: By assumption, there exists B such that $|a_n x_0^n| \le B$ for all n. Now let $r = |x_0|$. Then $|x_0/r| = 1$ so certainly $|x_0/r| \le 1$ and $|a_n| \le B/|x_0^n| = B/r^n$.

Now the point of these rather trivial observations is that we see that if $|x| < |x_0|$ then $\sum a_n x^n$ converges absolutely by comparison to the geometric series

$$\sum \left(\frac{x}{r}\right)^n$$

since $|a_n x^n| < B|x/r|^n$.

Now notice in particular that this theorem applies if $\sum a_n x_0^n$ converges, since in that case certainly $\lim_{n\to\infty} |a_n x_0^n| = 0$. ☑

If we turn this proposition around, we can see that it actually gives even more than it seems to.

Corollary. If $\sum a_n x_1^n$ diverges or converges conditionally for $x = x_1$, then it diverges for all x with $|x| > |x_1|$.

PROOF: If $|x| > |x_1|$ then $\sum a_n x^n$ can't converge, because otherwise by the Proposition (applied with x playing the role of x_0), $\sum a_n x_1^n$ would converge absolutely, contrary to the assumption. ☑

Radius of Convergence

We started out by asserting that if $\sum a_n x^n$ converges conditionally for a certain value x_0 of x, or if $\sum a_n x_0^n$ diverges but $\lim_{n\to\infty} |a_n x^n| < \infty$ then the series $\sum a_n x^n$ converges absolutely for all x with $|x| < |x_0|$ and diverges for all x with $|x| > |x_0|$. Clearly this assertion is included in the statements of the preceding Proposition and its Corollary.

In fact, though, we get an even more general result. What we see from the proposition and its corollary is that the set of values x where $\sum a_n x^n$ converges must be a interval (whether open, closed, or half-open) centered at the origin. The only exceptions are the cases when the series converges for all real numbers x, and the case where it diverges for all x except $x = 0$.

In other words:

Theorem. For a power series $\sum a_n x^n$, there are only three possibilities.

(1) The series converges for all values of x.

(2) The series converges only when $x = 0$.

(3) There exists a positive number R such that $\sum a_n x^n$ converges absolutely whenever $|x| < R$ and diverges whenever $|x| > R$.

In case (3), the number R is called the **radius of convergence** of the series. In case (1), we say that the radius of convergence is ∞, and in case (2) we say that it is zero.

If $\sum a_n x^n$ has a finite non-zero radius of convergence R, then the convergence of the series is roughly the same as the convergence of the series $\displaystyle\sum_0^\infty \frac{x^n}{R^n}$, which is a simple variation on the Geometric Series. The only difference is that $\sum a_n x^n$ may converge when $x = R$ or $x = -R$ or both, whereas $\displaystyle\sum_0^\infty \frac{x^n}{R^n}$ diverges for both these values.

If R is the radius of convergence for $\sum a_n x^n$, then $\lim_{n\to\infty} |a_n| R^n$ can be zero, infinity, or any positive number, or may not exist at all.

Examples of these four possibilities are as follows:

$$1 + x + x^2 + x^3 + \cdots \qquad a_n = 1, \quad R = 1, \quad \lim_{n\to\infty} a_n R^n = 1$$

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots \qquad a_n = n, \quad R = 1, \quad \lim_{n\to\infty} a_n R^n = \infty$$

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad a_n = \frac{1}{n}, \quad R = 1, \quad \lim_{n\to\infty} a_n R^n = 0$$

$$x + 3x^3 + 5x^5 + 7x^7 + \cdots \qquad a_n = \begin{cases} n \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even} \end{cases} \qquad R = 1, \quad \lim_{n\to\infty} a_n R^n \text{ does not exist.}$$

Suppose now that $\lim_{n\to\infty} |a_n|R^n$ does exist (or is infinity). The nice thing is that, according to the Proposition above, if $|x| > R$, then $\lim_{n\to\infty} |a_n x^n| = \infty$, which makes it really easy to see that the series diverges in this case. Obviously if $|x| < R$ then $\lim_{n\to\infty} a_n x^n = 0$ since in this case $\sum a_n x^n$ converges. Thus R can be characterized as the dividing point between those $x > 0$ such that $\lim_{x\to\infty} a_n x^n = 0$ and those such that $\lim_{n\to\infty} |a_n x^n| = \infty$.

It would be simplistic to say that as n approaches infinity, $|a_n|$ becomes roughly (or "asymptotically") comparable to a constant multiple of $(1/R)^n$. However what is true is that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = R$$

or else the limit is undefined. It is also true that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R$$

if the limit exists at all. These two formulas make it fairly easy to compute the radius of convergence for most power series.

The most frequent case where the two limits above do not exist is when the series skips powers of x. For instance the series for the sine function contains only odd powers of x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Clearly, then, the above two limits do not exist. One can get around this however by re-writing the series as

$$x\left(1 - \frac{x^2}{3!} + \frac{(x^2)^2}{5!} - \frac{(x^2)^3}{7!} + \cdots\right)$$

and then regarding what's inside the parentheses as a series in the variable x^2. If one writes b_n for the coefficients of this series (i. e. $b_0 = 1$, $b_1 = -1/3!$, $b_2 = 1/5!$, etc.) then one gets a radius of convergence for this series as $S = 1/\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{b_n}$. Thus the original series converges for $|x^2| < S$, so the radius of convergence for the original series is $\sqrt{S}$. (A fairly easy calculation shows that $S = \infty$ in this case, so the series for $\sin x$ converges for all values of x.)

Use of Complex Numbers

In the preceding, we have taken it for granted that the variable x and the coefficients a_i are real numbers. However the logic used applies just as well when these are complex.

Most students have probably seen complex numbers somewhere before (although probably without a convincing explanation as to why anyone should care about them!) Complex numbers have the form $c + di$, where c and d are real numbers and i is the square root of -1. Of course -1 **doesn't have** a square root. This is why complex numbers are not real!

We define the absolute value of a complex number $c + di$ as

$$|c + di| = \sqrt{c^2 + d^2}\,.$$

Geometrically, this means that $|c+di|$ is the length of the hypotenuse of a right triangle whose legs are $|c|$ and $|d|$. From this it is clear that $|c|<|c+di|$, $|d|<|c+di|$ and

$$|c+di|<|c|+|d|.$$

For an infinite series $\sum c_k + id_k$ we define the **real part** as the series $\sum c_k$ and the imaginary part as $i\sum d_k$. We say that the series **converges absolutely** if the corresponding absolutely value series $\sum |c_k+id_k|$ converges.

Since $|c_k|<|c_k+id_k|$ and $|d_k|<|c_k+id_k|$, the comparison test shows that if $\sum|c_k+id_k|$ converges, then so do $\sum|c_k|$ and $\sum|d_k|$. And since $|c_k+id_k|<|c_k|+|d_k|$, if follows that if $|c_k|$ and $\sum|d_k|$ both converge, then so does $\sum|c_k+id_k|$. Thus a series $\sum c_k+id_k$ converges absolutely if and only if both the real and imaginary parts converge absolutely. And from this we see that a complex series which converges absolutely does in fact converge.

Now all the logic used above for real power series applies just as well to a complex power series $\sum(c_k+id_k)x^k$ and shows that

Theorem. For a complex power series $\sum(c_n+id_n)x^n$, there are only three possibilities.

(1) The series converges for all values of x.

(2) The series converges only when $x=0$.

(3) There exists a positive number R such that $\sum a_nx^n$ converges absolutely whenever $|x|<R$ and diverges whenever $|x|>R$.

In case (3), the number R is called the **radius of convergence** of the series.

It is usual to think of a complex number $c+id$ as corresponding to the point in the Euclidean plane with coordinates (c,d). In terms of this representation, we see that the power series converges for all points x strictly inside the circle around 0 with radius R, and diverges for all points strictly outside that circle. For points actually on the circle itself, convergence is a delicate question.

In case (1), we say that the radius of convergence is ∞, and in case (2) we say that it is zero.

Why Use Complex Numbers?

One might wonder why anyone would ever want to do calculus using complex numbers. There are three reasons why this can be worthwhile.

In the first place, even though complex numbers do not occur in most problems from ordinary life, there are many places in science where they are quite useful. This is especially true in electrical engineering, where they are a standard tool for the study of AC circuits.

Secondly, complex numbers enable one to considerably simplify certain sorts of calculations even when one only really cares about real numbers.

And third, in a lot of ways calculus simply makes more sense when one uses complex numbers and it becomes a much more beautiful subject.

Although there is a sizable literature on the philosophy of mathematics (very little of which I have read), I don't know of anyone who has ever tried to discuss precisely what we mean when we talk about "beauty" in mathematics. But it seems to me that one of the things that makes a particular piece of mathematics beautiful is the existence of an unexpected orderliness — a nice pattern where one would have expected only chaos. In fact, I think that one of the things that attracts people to the study of mathematics is that in many parts of mathematics one finds a world that is orderly, a world where things *make sense* in a way that things in the real world rarely do. (And yet the study of this unreal, orderly mathematical world, which is almost like a psychotic fantasy that an obsessive-compulsive personality might have come up with, can produce very useful applications in terms of the real world. This is a fascinating paradox.)

When one first thinks about the set of points at which a power series would converge, one might imagine that it could have any conceivable form, or even be completely formless. To me, the fact that this set of points in fact forms the interior of a circle — geometry's most perfect figure — is beautiful.

But furthermore, when one looks at complex numbers one discovers that the radius of convergence for a power series is exactly what it needs to be. The radius of convergence for a power series will always be as big as it possibly can in order to avoid having any singularities of corresponding function within the circle of convergence. (Unfortunately, though, I don't know any proof of this simple enough to include here.)

If we look at the expansion for $1/(1-x)$ as a series in powers of x, for instance, we find that the radius of convergence is 1. This makes sense: it couldn't be any larger than 1 since the function $1/(1-x)$ blows up at $x=1$.

Likewise, the radius of convergence for the expansion of $\ln(1+x)$ as a series in powers of x is 1, which is just small enough to avoid the point $x=-1$ where the function blows up.

On the other hand, if we think only of real numbers, the function $\tan^{-1} x$ has no bad points. It is continuous, differentiable, and even analytic for all real numbers x. So then why does its power series

$$1-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

have 1 as its radius of convergence rather than ∞? As long as we only consider real values for x, it doesn't make any sense. But if we look at complex values it does. Remember that the radius of convergence for $\tan^{-1} x$ will be the same as the radius of convergence for its derivative. And the derivative of $\tan^{-1} x$ is $1/(1+x^2)$. Now $1/(1+x^2)$ can never blow up as long as x is a real number, since in this case x^2 is positive and so $1+x^2\neq 0$. But if $x=i$ (where $i=\sqrt{-1}$), then $1+x^2=0$ and so $1/(1+x^2)$ blows up. And once we figure out how to define $\tan^{-1} x$ when x is a complex number, we will discover that this function also blows up when $x=i$. The radius of convergence for the power series $\tan^{-1} x$ is thus as big as it possibly can be and yet still exclude the singular point $x = i$.

Differentiation and Integration of Power Series

If we take the Geometric Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

and differentiate it in the obvious way, we get an equation

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots .$$

(Note that the differentiation of the left hand side produces two minus signs which cancel each other.) On the other hand, if we integrate the geometric series in the obvious way we get an equation

$$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{x} + \frac{x^4}{4} + \cdots \qquad (C = 0)$$

(where C is the constant of integration, which in this case must be equal to $\ln 1 = 0$). These two results are actually correct. In fact, simple algebra shows that

$$\begin{aligned}
&(1-x)(1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n) \\
&\quad = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n \\
&\qquad\quad -x - 2x^2 - 3x^3 - \cdots \qquad - nx^n - (n+1)x^{n+1} \\
&\quad = 1 + x + x^2 + x^3 + \cdots \qquad + x^n - (n+1)x^{n+1}
\end{aligned}$$

and so

$$\begin{aligned}
(1-x)^2(1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n) &= (1-x)(1 + x + x^2 + \cdots + x^n - (n+1)x^{n+1}) \\
&= 1 - (n+2)x^{n+1} - (n+1)x^{n+2} .
\end{aligned}$$

Since $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$, it follows that

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \lim_{n\to\infty} \frac{1 - (n+2)x^{n+1} - (n+1)x^{n+2}}{(1-x)^2} = \frac{1}{(1-x)^2}$$

for $|x| < 1$ and the series diverges for $|x| \geq 1$.

The correctness of the second formula can be verified if one knows that Taylor Series expansion of the natural logarithm function, except that in this case the series also converges for $x = -1$ by the Alternating Series Test.

It may seem quite obvious that, in general, if

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^2 + \cdots$$

then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots .$$

But this, although true, is not as obvious as it might seem. Because the notation makes an infinite series look like merely a process of adding an enormous number of things up, it is tempting to assume that all the things that work for algebraic addition will also work for infinite series. But in fact this is not always the case, and there do exist infinite series where differentiation does not yield the result one would expect it to.

This is shown by the fact that the series for $\arctan x$,

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots ,$$

converges for $-1 \leq x \leq 1$, whereas the derivative series,

$$1 - x^2 + x^4 - x^6 + \cdots$$

(which represents the function $1/(1+x^2)$ does not converge for $x = \pm 1$.

A less simplistic example is the series

$$\cos x + \frac{1}{4}\cos 8x + \frac{1}{9}\cos 27x + \cdots + \frac{1}{n^2}\cos n^3 x + \cdots .$$

(Note that this is not a power series.) Comparison to the series $\sum \frac{1}{n^2}$ shows that this converges absolutely for all x. But if we differentiate it term by term, we get

$$-\sin x - 2\sin 8x - 3\sin 27x - \cdots - n\sin n^3 x - \cdots$$

which seems to clearly diverge for most values of x. (The derivative series clearly converges when x is a multiple of π, since all the terms are then 0. But when $x = 2k\pi + \frac{\pi}{4}$, for instance, where k is an integer, the series looks like

$$-\sin\frac{\pi}{4} - 2\sin\frac{8\pi}{4} - 3\sin\frac{27\pi}{4} - \ 4\sin\frac{64\pi}{4} - 5\sin\frac{125\pi}{4} - \cdots - n\sin\frac{n^3\pi}{4} + \cdots$$

$$= -\frac{\sqrt{2}}{2} - 0 - 3\cdot\frac{\sqrt{2}}{2} - 0 + 5\cdot\frac{\sqrt{2}}{2} - 0 + \cdots ,$$

which clearly diverges since we keep seeing larger and larger terms: for odd n we have $\pm\frac{n\sqrt{2}}{2}$. This shows that the domain of convergence for a series which is not a power series need not be an interval.)

It turns out, however, that **in the case of power series**, differentiating and integrating in the obvious fashion does always work in the way one would hope, the only exception being at the endpoints of the interval of convergence.

To see why this is true, let's start by looking at the radius of convergence of the differentiated series.

Recall that we observed above that the radius of convergence R for a power series $\sum a_n x^n$ is the point on the positive number line that separates the set of x such that $\lim_{n\to\infty} a_n x^n = 0$ from those such that $\lim_{n\to\infty} |a_n x^n| = \infty$. In particular, for $|x| > R$ then $\lim_{n\to\infty} |a_n x^n| = \infty$. But then

$$\lim_{n\to\infty} |na_n x^{n-1}| = \frac{1}{|x|}\lim_{n\to\infty} |na_n x^n| \geq \frac{1}{|x|}\lim_{n\to\infty} |a_n x^n| = \infty ,$$

so clearly the differentiated series $\sum na_n x^{n-1}$ diverges.

In particular, if $R=0$ then the differentiated series always diverges for $x \neq 0$.

Now suppose for the moment that the power series $\sum a_n x^n$ has a radius of convergence R which is neither zero nor infinity. Then we have seen above that the tweaked form of the comparison test can be used to see that, as respects convergence, the series $\sum a_n x^n$ behaves exactly like the Geometric Series

$$\sum \left(\frac{x}{R}\right)^n$$

for $|x|<R$ (where both series converge) and $|x|>R$ (where both series diverge). (The behavior of the series when $x=\pm R$ is a much more delicate matter and varies depending on the specific series.) But the comparison test can then also be used to show that the series $\sum n a_n x^{n-1}$ and

$$\sum n \left(\frac{x}{R}\right)^{n-1}$$

have the same radius of convergence. But the radius of convergence for $\sum n \left(\dfrac{x}{R}\right)^{n-1}$ has already been shown to be R.

The reasoning here is simplest if, as is often the case, $\lim_{n\to\infty} |a_n| R^n < \infty$. (This would always be the case, for instance, if the power series converges at either of the two endpoints of its interval of convergence.) From this we get

$$\lim_{n\to\infty} \frac{|n a_n x^{n-1}|}{|n(x/R)^{n-1}|} = \frac{1}{R} \lim_{n\to\infty} |a_n R^n| < \infty$$

(recall that we are temporarily assuming that $R \neq 0, \infty$), so the limit comparison test shows that $\sum n a_n x^{n-1}$ converges absolutely whenever $\sum n \left(\dfrac{x}{R}\right)^{n-1}$ does. Since it was shown above that $\sum n \left(\dfrac{x}{R}\right)^{n-1}$ converges absolutely when $|x/R|<1$ and diverges for $|x/R|>1$, it follows that

$$0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

converges for $|x|<R$. Thus, in the case when $0<R<\infty$ and $\lim_{n\to\infty} |a_n| R^n < \infty$, the differentiated series has the same radius of convergence as the original series. (If $0 < \lim_{n\to\infty} |a_n| R^n$, note also that clearly $\sum a_n x^n$ diverges for $x=\pm R$, and thus $\sum n a_n x^{n-1}$ also diverges, since $\sum n a_n x^{n-1} = \dfrac{1}{x} \sum n a_n x^n$ and $|n a_n| > |a_n|$.)

Now if $\lim_{n\to\infty} |a_n| R^n = \infty$, then we have to use a little more finesse to prove convergence for $|x|<R$. If $|x|<R$ then choose r between $|x|$ and R:

$$|x| < r < R\,.$$

Then, as noted earlier, $\lim_{n\to\infty} a_n r^n = 0$, and we can see that $\sum n a_n x^{n-1}$ converges absolutely by using the limit comparison test with respect to

$$\sum n \left(\frac{x}{r}\right)^{n-1}.$$

This reasoning also shows that the differentiated series

$$\sum a_n x^n$$

converges absolutely for all x in case $R=\infty$.

This finishes the proof that the differentiated series has the same radius of convergence as the original series.

Since the original series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

can be obtained by differentiating the integrated series

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots ,$$

the above shows that the original series and the integrated series also have the same radius of convergence.

It sometimes happens, however, that the original series converges at one or both endpoints of the interval of convergence but the differentiated series does not. For instance, as previously mentioned, the series for the $\tan^{-1} x$,

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

turns out to have 1 as its radius of convergence. It converges conditionally at $x = \pm 1$. The differentiated series

$$1 - x^2 + x^4 - x^6 + \cdots$$

is simply the Geometric Series in the variable $-x^2$, and has the same radius of convergence, viz 1, but does not converge at either $x = 1$ or $x = -1$.

When $R = 0$ or $R = \infty$, the endpoints are not a problem, of course, since the interval of convergence has no endpoints.

Now that we've seen that the differentiated power series has the same radius of convergence as the original one, it remains to be shown that if

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

then

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

does in fact equal $f'(x)$ for those values of x where both series converge.

There's a very easy (but flawed) explanation here Suppose we start with a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

According to the standard formula for Taylor series,

$$a_n = \frac{f^{(n)}(0)}{n!} .$$

Now apply the same standard formula to compute $f'(x)$:

$$\begin{aligned} f'(x) &= f'(0) + (f')'(0)x + \frac{(f')^{(2)}(0)}{2!}x^2 + \frac{(f')^{(3)}(0)}{3!}x^3 + \cdots \\ &= f'(0) + f^{(2)}(0)x + \frac{f^{(3)}(0)}{2!}x^2 + \frac{f^{(4)}(0)}{3!}x^3 + \cdots + \frac{f^{(n+1)}(0)}{n!}x^n + \cdots . \end{aligned}$$

But from the formula above, we have $f^{n+1}(0) = (n+1)!\, a_{n+1}$. Substituting this, we get

$$\begin{aligned} f'(x) &= a_1 + 2!\, a_2 x + \frac{3!\, a_3}{2!}x^2 + \cdots \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots \end{aligned}$$

where we have used the identity $(n+1)! = (n+1)\cdot n!$. But this series expansion for $f'(x)$ is exactly the one we get by differentiating the original series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

in the obvious way.

Unfortunately, however, this simple explanation is flawed. We have made two assumptions which, although true, are as yet unproved. In the first place, we have assumed that if a function has a Taylor series, then the coefficients for that series must necessarily be given by the standard formula

$$a_n = \frac{f^{(n)}(0)}{n!} .$$

And second, we've assumed that if $f(x)$ has a Taylor series and that Taylor series does in fact converge to $f(x)$, then the same will be true of $f'(x)$.

What's It All For?

Why do we care about infinite series in the first place? And in particular, why should we care about showing that an infinite series converges if we don't know what it converges **to**?

The value of infinite series is exactly the same as the value of algebraic formulas. Namely, they can be used to **calculate** things and also to **define** things.

For example, Taylor Series give us the only practical way of computing such common functions as $\sin x$, $\ln x$, or e^x. Calculators and computer software all use accelerated forms of Taylor series to compute these function.

This process has theoretical as well as practical significance. There are many functions which cannot be described by simple algebraic formulas such as $f(x) = \dfrac{x^3 - 8x + 2}{x^2 + 5}$, or even $f(x) = \sqrt{8x^2 + 7}$. We are often not lucky enough to be able to find some geometric relationship to describe such functions, as is the case with the trig functions, or to be able to describe them by simple integrals, as can be done for $\ln x$. Power series give one a much more far-reaching type of formula that can be used to describe a large class of functions, which include almost all those which are likely to arise in practical applications.

The solutions to linear ordinary differential equations can usually be expressed in the form of Taylor Series, whereas in many cases they cannot be described in terms of common functions such as $\sin x$ or e^x.

Likewise, Fourier Series give one of the primary means for solving partial differential equations.

The Power Series Method for Differential Equations

The use of infinite series to solve differential equations is one of the reasons why it's so important to know that the obvious method for differentiating and integrating power series is valid.

In using power series to solving an ordinary differential equation, one starts by assuming that the solution function can be written in the form of a power series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

and then substitutes this expression into the differential equation in order to solve for the coefficients a_i.

Not only can one use this to find solutions that cannot otherwise be easily written down, but the same trick can be used to find the Taylor series expansions for many familiar functions. The sine function and cosine functions, for instance, satisfy the differential equation

$$f''(x) = -f(x)\,.$$

They are further characterized by the fact that, for the sine function, $f(0) = 0$ and $f'(0) = 1$, whereas for the cosine function, $f(0) = 1$ and $f'(0) = -1$. It is easy to use this differential equation to derive the familiar Taylor series expansions for $\sin x$ and $\cos x$.

In using this method, one should note that a function may satisfy several different differential equations. Some of these may be easy to work with while others may not. For instance, if we wanted to use the power series method to derive the geometric series, it would be natural to start with the observation that the function $f(x) = 1/(1-x)$ satisfies the differential equation

$$f'(x) = f(x)^2\,.$$

However substituting the power series into this equation is rather awkward, since on the left hand side one needs to square the power series, which is an ugly process. However one can also note that the function $f(x) = 1/(1-x)$ satisfies the equation

$$(1-x)f'(x) = (1-x)\,\frac{1}{(1-x)^2} = f(x)$$

(compare this to the binomial series, below). From this, the derivation of the geometric series is easy:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \qquad + a_{k-1}x^{k-1} \qquad + a_k x^k \quad + a_{k+1}x^{k+1} + \cdots \\
f'(x) &= a_1 + 2a_2 x + a_3 x^2 + \cdots \qquad + k a_k x^{k-1} + (k+1)a_{k+1}x^k + \cdots \\
-xf'(x) &= \quad - \; a_1 x - 2a_2 x^2 - \cdots - (k-1)a_{k-1}x^{k-1} \qquad - k a_k x^k - \cdots \,,
\end{aligned}$$

which, together with the initial condition $a_0 = f(0) = 1$, yields

$$\begin{aligned}
a_1 &= a_0 \\
2a_2 - a_1 &= a_1 \\
&\cdots \\
(k+1)a_{k+1} - k a_k &= a_k
\end{aligned}$$

and if then follows easily that

$$a_{k+1} = a_k = \cdots = a_2 = a_1 = a_0 = 1\,.$$

However one can derive the geometric series even more easily by noticing that the function $f(x) = 1/(1-x)$ satisfies the algebraic equation $(1-x)f(x) = 1$. Leaving this as a very easy exercise for the reader, we will use essentially the same trick to derive the series for the arctangent function. The function $f(x) = \tan^{-1} x$ satisfies the differential equation

$$(1+x^2)f'(x) = 1\,.$$

From this, we get the following:

$$\begin{aligned}
f(x) &= a_0 + a_1 x \;\; + a_2 x^2 + \cdots \qquad + a_{k-1}x^{k-1} \qquad + a_k x^k + a_{k+1}x^{k+1} + \cdots \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (k-1)a_{k-1}x^{k-2} + k a_k x^{k-1} + (k+1)a_{k+1}x^k + \cdots \\
x^2 f'(x) &= \qquad a_1 x^2 + 2a_2 x^3 + \cdots \qquad + (k-1)a_{k-1}x^k + \cdots \,.
\end{aligned}$$

The differential equation $(1+x^2)f'(x) = 1$ together with the initial condition $f(0) = 0$ then yields

$$\begin{aligned}
a_1 &= 1 \\
2a_2 &= 0 \\
3a_3 + a_1 &= 0 \\
&\cdots \\
(k+1)a_{k+1} + (k-1)a_{k-1} &= 0
\end{aligned}$$

from which we easily conclude that $a_k = 0$ for all even integers k and that for odd k, $a_k = \pm\dfrac{1}{k}$, where the signs alternate. I. e.

$$\tan^{-1} x = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,.$$

This method for solving differential equations is not universally useful, though. The Taylor Series expansion for $\tan x$ is very difficult to compute using the standard formula, since the derivatives quickly become quite complicated. For instance,

$$\frac{d^2}{dx^2}\tan x = 2\sec^2 x \tan x$$

$$\frac{d^3}{dx^3}\tan x = 4\sec^2 x \tan^2 x + 2\sec^4 x\,,$$

and from there on, things get even worse.

If we try the differential equation approach, we can notice that if $f(x) = \tan x$ then $f'(x) = \sec^2 x = \tan^2 x + 1$, so that $\tan x$ satisfies the differential equation

$$f'(x) = f(x)^2 + 1$$

with the initial condition $f(0) = 0$. However this differential equation is not "linear," since the unknown function occurs in it squared. This means that if one tries writing

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

and substituting this into the differential equation, one has to square the power series, which is a rather ugly process. (One has to include all the cross-products, getting

$$f(x)^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + (2a_0a_3 + 2a_1a_2)x^3 + (2a_0a_4 + 2a_1a_3 + a_2^2)x^4 + \cdots .)$$

For the case $f(x) = \tan x$, one has $a_0 = f(0) = 0$, $a_1 = f'(0) = 1$. Thus from the differential equation $f'(x) = 1 + f(x)^2$ one gets

$$\begin{aligned}
a_1 &= 1 \\
2a_2x &= 0 \\
3a_3x^2 &= a_1x^2 = x^2 \\
4a_4x^3 &= 2a_1a_2x^3 = 2a_2x^3 \\
5a_5x^4 &= (2a_1a_3 + a_2^2)x^4 = (2a_3 + a_2^2)x^4 \\
&\cdots
\end{aligned}$$

which implies $a_2 = 0$, $a_3 = \frac{1}{3}$, $a_4 = 0$, and $a_5 = 2/15$.

In using the differential equation method to derive the Taylor series for a function, one starts out from the *assumption* that the function in question is **analytic**, i. e. that it can be represented by some power series. This amounts to a gap in the logic if one's interest is in deriving everything rigorously from the ground up. However often one's concern is more practical: one knows as a matter of fact (or is willing to believe) that the function is analytic, and one just wants a practical method of figuring out what the Taylor series is.

In any case, one can say this. If one substitutes a generic expression of a power series into a differential equation and solves for the unknown coefficients, and if the resulting power series does converge at least in some interval, then it will in fact represent a function which satisfies the given differential equation. Therefore the only possible thing that might go wrong is that there is in fact more than one function satisfying the given differential equation with given intial conditions, and that the power series one has found gives *one* of these functions, but not the one actually desired.

Anyone who has much experience with differential equations is likely to take it for granted that this will not in fact occur. One knows that there are existence and uniqueness theorems which guarantee that a differential equation with appropriate initial conditions almost always has a unique solution. A check with a text on differential equations, however, will show that these existence and uniqueness theorems do depend on the function in question satisfying certain conditions which, although seeming fairly innocuous, can conceivably fail.

Bad examples are not easy to come by, but consider the function

$$e^{-1/x^2}.$$

(This is almost the only bad example that can be written down in any convenient form, at least as far as I know.) This function is technically undefined at 0, but assigning the value $f(0)=0$ will make it continuous, since

$$\lim_{x\to 0} e^{\frac{-1}{x^2}} = e^{-\infty} = 0\,.$$

(One can now see the careful logic of the way this function is defined. One needs the minus sign in the exponent in order to make the function go to 0 when x goes to 0. Using $e^{-1/x}$ would not have worked, because as x approaches 0 from the left, one would then get $e^{+\infty}=\infty$.)

Now let $g(x)=e^{-1/x^2}$. Notice that

$$g'(x) = \frac{2}{x^3}\,e^{\frac{-1}{x^2}}\,,$$

so that $g(x)$ satisfies the differential equation

$$x^3 g'(x) = 2g(x)\,.$$

This equation is linear. (The fact that it involves x^3 is not important. What's important is that the function itself is not squared or cubed.)

Now suppose that $g(x)$ has a Taylor series expansion

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\,,$$

with $a_0 = g(0) = 0$. Substituting this into the differential equation yields

$$a_0x^3 + a_1x^4 + a_2x^5 + + \cdots = 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + 2a_5x^5 + \cdots\,.$$

Comparing coefficients, we immediately see that $a_0=0$, $a_1=0$, and $a_2=0$. It then follows that $2a_3=a_0=0$, and so $2a_4=a_1=0$, and in fact, one can eventually show that $a_k=0$ for all k.

We have thus managed to prove that $g(x)=0$ for all x. But this result has to be fallacious, since raising e to any finite power can never produce 0 (although one can get very close by taking x extremely negative).

The fallacy here consists not in the way we have manipulated the power series, but in the assumption that e^{-1/x^2} can be represented by a power series in the first place . Unlike most of the functions one finds in calculus books (as well as physics and chemistry and biology books), this function is not **analytic** at 0. The reason for this is that the exponential function has a rather complicated sort of singularity at ∞, and by choosing $-1/x^2$ we have managed to transfer this singularity to the origin, with the result that the function fails to be analytic there, despite being differentiable. (In fact, $g^{(k)}(0)$ is defined and equal to 0 for all k.)

The differential equation

$$x^3 y' = 2y$$

satisfied by the function $y = e^{-1/x^2}$ does not satisfy the conditions given in existence and uniqueness theorem. (If we write this equation in standard form as

$$y' = \frac{2y}{x^3},$$

we see that the right-hand side is not continuous at $x = 0$.) Indeed there are at least two different solutions to this equation both satisfying the initial condition $y(0) = 0$, namely $y = e^{-1/x^2}$ and $y = 0$.

To see why this happens, let's go through the process of solving this differential equation. The technique for "separable" equations applies, and we get a solution by going through the following sequence of steps.

$$\begin{aligned} x^3 y' &= 2y \\ \frac{y'}{y} &= 2x^{-3} \\ \ln y &= -x^{-2} + \ln c \\ y &= e^{-1/x^2 + \ln c} = ce^{-1/x^2} \end{aligned}$$

If we assign the arbitrary constant here the value $c = 0$, we get $y(0) = 0$. However if we assign the constant the value $c = 1$, we also get $y(0) = 0$, since $e^{-\frac{1}{0}} = e^{-\infty} = 0$. In fact, no matter what value we assign to c, it will always be true that $y(0) = 0$. Thus the differential equation $x^3 y' = 2y$ with initial condition $y(0) = 0$ has an infinite number of different solutions.

And in fact, for this equation existence also breaks down. Because, as we have seen, any possible solution to $x^3 y' = 2y$ will have the property that $y(0) = 0$. Thus there is no solution, for instance, to the differential equation $x^3 y' = 2y$ with the initial condition $y(0) = 5$.

The Binomial and Negative Binomial Series

Another interesting example of using Taylor series to solve differential equations is the derivation of binomial series. If we write

$$f(x) = (1 + x)^p,$$

where p is any real number (in particular, probably not a positive integer), then $f'(x) = p(1 + x)^{p-1}$, and it follows that

$$(1 + x)f'(x) = pf(x).$$

Let's now use the power series method to solve this differential equation. Assuming that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

we get

$$\begin{aligned} f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ xf'(x) &= \qquad a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots \end{aligned}$$

so that the equation $(1+x)f'(x) = pf(x)$ implies

$$\begin{aligned} & a_1 + (a_1 + 2a_2)x + (2a_2 + 3a_3)x^2 + \cdots \quad + (ka_k + (k+1)a_{k+1})\, x^k + \cdots \\ = {} & pa_0 \qquad + pa_1 x \qquad + \qquad \cdots \qquad\qquad + pa_k x^k + \cdots . \end{aligned}$$

Furthermore, from the fact that $f(0) = (1+0)^p = 1$, we get

$$a_0 = 1 .$$

We can now solve for the a_i recursively be equating the coefficients of the corresponding powers of x, as follows:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= pa_0 = p \\ a_2 &= \frac{(p-1)a_1}{2} = \frac{p(p-1)}{2} \\ a_3 &= \frac{(p-2)a_2}{3} = \frac{p(p-1)(p-2)}{3 \cdot 2} \\ &\ldots \\ a_k &= \frac{(p-(k-1))}{k} = \frac{p(p-1)(p-2)\cdots(p-(k-1))}{k!} . \end{aligned}$$

It is customary to define the **binomial coefficients** to be

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-(k-1))}{k!} .$$

(Here p need not be either positive or an integer.) With this notation, the binomial series can be written as

$$\boxed{(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \cdots + \binom{p}{k}x^k + \cdots .}$$

It is easy to see that if p is not a positive integer, this series has 1 as a radius of convergence, whereas when p is a positive integer all the terms after the p^{th} one vanish, so the series becomes a simple polynomial and therefore converges (so to speak) for all x.

As a variation on the binomial series, consider what happens when we differentiate the geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = 2\cdot 1 + 3\cdot 2x + 4\cdot 3x^2 + 5\cdot 4x^3 + 6\cdot 5x^4 + \cdots$$

$$\frac{3\cdot 2}{(1-x)^4} = \frac{d^3}{dx^3}\left(\frac{1}{1-x}\right) = 3\cdot 2\cdot 1 + 4\cdot 3\cdot 2x + 5\cdot 4\cdot 3x^2 + 6\cdot 5\cdot 4x^3 + \cdots$$

$$\frac{4\cdot 3\cdot 2}{(1-x)^5} = \frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = 4\cdot 3\cdot 2\cdot 1 + 5\cdot 4\cdot 3\cdot 2x + 6\cdot 5\cdot 4\cdot 3x^2 + \cdots$$

Continuing in this fashion, we find that, in general,

$$\frac{(r-1)!}{(1-x)^r} = (r-1)! + r(r-1)\cdots 2\, x + (r+1)r\cdots 3\, x^2 + \cdots + (r+k-1)\cdots(k+1)\, x^k + \cdots$$

Dividing through, and applying the formula for the binomial coefficients, and using the identity

$$\binom{r+k-1}{r-1} = \binom{r+k-1}{k},$$

we get

$$\begin{aligned}(1-x)^{-r} &= 1 + \binom{r}{r-1}x + \binom{r+1}{r-1}x^2 + \cdots + \binom{r+k-1}{r-1}x^k + \cdots .\\ &= 1 + \binom{r}{1}x + \binom{r+1}{2}x^2 + \cdots + \binom{r+k-1}{k}x^k + \cdots .\end{aligned}$$

This is the **negative binomial series**. It has important applications in probability theory and combinatorics.

The function $(1-x)^{-r}$ can also be expanded using the standard binomial series. One gets

$$(1-x)^{-r} = 1 + \binom{-r}{1}(-x) + \binom{-r}{2}(-x)^2 + \binom{-r}{3}(-x)^3 + \cdots .$$

But a function can have only one power series expansion, since this expansion must be given by the standard Taylor series formula. Therefore it must be true that these two expansions for $(1-x)^{-r}$ are in fact identical, in other words, that

$$\binom{-r}{k}(-x)^k = \binom{r+k-1}{k}x^k$$

for all k.

In fact, it's easy to see this. For instance

$$\binom{-r}{3}(-x)^3 = \frac{(-r)(-r-1)(-r-2)}{3!}(-1)^3x^3$$

$$= \frac{(-1)^3 r(r+1)(r+2)}{3!}(-1)^3x^3 = (-1)^6\binom{r+2}{3}x^3 = \binom{r+3-1}{3}x^3 .$$

The Trig Functions and the Exponential Function

There are only a handful of functions whose Taylor series has a convenient formula. Fortunately, these are precisely the functions which occur most frequently in mathematics and its applications. One might wonder whether this is merely a piece of good luck, or whether the fact that functions such as the sine and cosine, the exponential and the logarithm are useful in so many different ways might in some way be a consequence of the fact that they have very nice Taylor series, or at least be closely related to that fact.

One first learns about the sine and cosine in connection with the measurement of triangles. At first, then, it seems that these functions would be primarily of interest to surveyors, and other people concerned with practical applications of simple geometry. But as one learns more science, and more calculus, one finds the sine and the cosine functions coming up over and over again in contexts with no apparent relationship with triangles. For instance if one plots the length of the day as a function of the time of year, one finds that the resulting curve has the shape of the sine function (or cosine function). The voltage produced by a generator is also described by a sine function, as if the output of an electornic oscillator constructed by using a simple LC circuit. And even simple harmonic motion, such as the motion of a weight attached to a spring, is described by the sine function.

Some of these facts can be explained on the basis of geometry. But it seems more enlightening to see them all as consequences of the fact that if $y = \sin x$ or $y = \cos x$, then $y'' = -y$. The theory of differential equations further tells us that if $y(x)$ is any function satisfying the differential equation $y'' = -y$, then $y(x)$ can be written in the form $A \sin x + B \cos x$, where A and B are constants.

Likewise, the wide ranging importance of the exponential function (for instance in describing population growth and radioactive decay) can be seen as largely due to the fact that if $y(x) = Ce^x$ (where C is a constant), then y satisfies the differential equation $y' = y$.

Now there is a certain similarity in this respect between the sine and cosine functions and the exponential function. To try and see this more clearly, consider the function $f(x) = e^{-x}$. This function satisfies the differential equation $f' = -f$, which is very much like the equation $f'' = -f$ satisfied by the sine and cosine functions.

From this point of view, the exponential function and the sine and cosine functions, which at first seem so different from each other, seems on some level to be very similar. And, in fact, when one looks at the Taylor series expansions, it almost looks like the sine function and the cosine functions could be fit together as the two halves of the exponential function.

$$\begin{array}{rlllllll}
e^x = 1 & +x & +\dfrac{2^x}{2} & +\dfrac{x^3}{3!} & +\dfrac{x^4}{4!} & +\dfrac{x^5}{5!} & +\cdots \\
\cos x = 1 & & -\dfrac{2^x}{2} & & +\dfrac{x^4}{4!} & & -\cdots \\
\sin x = & x & & -\dfrac{x^3}{3!} & & +\dfrac{x^5}{5!} & & -\cdots \\
e^{-x} = 1 & -x & +\dfrac{2^x}{2} & -\dfrac{x^3}{3!} & +\dfrac{x^4}{4!} & -\dfrac{x^5}{5!} & +\cdots
\end{array}$$

However the minus signs don't quite work here. And, as in most parts of mathematics, the difficulty with the minus signs is not something which can be ignored or easily fixed.

Look at it this way: If $f(x) = \sin x$ or $f(x) = \cos x$, then $f''(x) = -f(x)$. The fact that differentiating twice reverses the sign of the function corresponds to the fact that in the Taylor series, there is a difference of 2 between the degrees of successive terms, and these have opposite signs (e. g.. $-x^2/2!$ and $+x^4/4!$ or $-x^3/3!$ and $+x^5/5!$). On the other hand, in the Taylor series for e^{-x}, positive terms and negative terms follow each other with a gap of only 1 between the degrees (e. g. $+x^2/2!$ and $-x^3/3!$), which corresponds to the fact that differentiating the function once reverses the sign.

We could also look at it this way: differentiating e^{-x} **once** results in a 180° reversal, and differentiating $\sin x$ or $\cos x$ **twice** results in a 180° reversal. One might be tempted then to conjecture that somehow or other differentiating $\sin x$ or $\cos x$ once results in a 90° rotation. And, in fact, differentiating $\sin x$ or $\cos x$ produces a phase-shift of $\pi/2$ in the corresponding graph.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x = \sin(x + \tfrac{\pi}{2}) \\
\frac{d}{dx}(\cos x) &= -\sin x = \cos(x + \tfrac{\pi}{2}) \\
\frac{d^2}{dx^2}(\sin x) &= -\sin x = \sin(x + \pi) \\
\frac{d^2}{dx^2}(\cos x) &= -\cos x = \cos(x + \pi)
\end{aligned}$$

However the thinking here is a little muddled, since we have started to interpret the phrase "180° reveral" in two different ways.

On some level this can be straightened out, but the explanation becomes much simpler if we allow the use of complex numbers. If we consider the function $g(x) = e^{ix}$, then differenting this function does in fact in some sense produce a 90° rotation: $g'(x) = ig(x)$. To see why it makes sense to say this, notice that

$$g''(x) = i^2 g(x) = -g(x)\,,$$

so that differentiating twice prduces a 180^{o} reversal.

Now $\sin x$ and $\cos x$ are two independent solutions to the differential equation $y'' = -y$. Since we now see that e^{ix} is also a solution to this differential equation, the general theory states that there should be constants A and B such that

$$e^{ix} = A\cos x + B\sin x\,.$$

Now admittedly, once we start using complex numbers in this way we are moving into uncharted territory, and we should no longer automatically assume that the old rules work. But we'll assume for the time being that they do (since I happen to know that they do!) and see if we can find a way of making this equation work. Doing some elementary manipulation, we get the following set of equations:

$$\begin{aligned} e^{ix} &= A\cos x + B\sin x \qquad \text{(by assumption!)} \\ 1 = e^0 &= A\cos 0 + B\sin 0 = A \\ ie^{ix} = \frac{d}{dx}\,e^{ix} &= \frac{d}{dx}(A\cos x + B\sin x) = -A\sin x + B\cos x \\ i = ie^0 &= -A\sin 0 + B\cos 0 = B\,. \end{aligned}$$

We then conclude that

$$\boxed{e^{ix} = \cos x + i\sin x\,.}$$

Now this is not a proof, and in fact *cannot* be a proof, because, so far, e^{ix} has not even been defined! What we have shown, though, is that **if** the basic principles from the theory of differential equations are going to apply to calculus when complex numbers are allowed, then $e^{ix} = \cos x + i\sin x$ is the only possible definition that will make sense.

As another way of seeing this, we can compute the Taylor series for e^{ix} and compare it to $\cos x + i\sin x$. Using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, etc., we get

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2} - i\,\frac{x^3}{3!} + \frac{x^4}{4!} + i\,\frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \\ i\sin x &= ix - i\,\frac{x^3}{3!} + i\,\frac{x^5}{5!} - \cdots \end{aligned}$$

These Taylor series expansions give a second justification (based on the assumption that the usual Taylor series expansion for the exponential function should continue to work when the variable is a complex number) for defining $e^{ix} = \cos x + i\sin x$. Once we accept this definition (and indeed we should), then it becomes possible to apply the exponential function to any complex number $x+iy$. In fact, if x and y are real numbers (or, in fact, even if they are complex!) we get

$$\boxed{e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)\,.}$$

Since this formula unifies the concepts of the sine, cosine, and exponential functions, it turns out to be a very handy even in many cases where one is ultimately interested only in real numbers. In fact, it is precisely for this reason that complex numbers are so useful in electronic circuit theory.

Since we have seen that the exponential function can be applied to complex values of the independent variable (although for a completely rigorous discussion one should consult a textbook on complex analysis), one might wonder whether the same is true for the sine and cosine (as in fact has already been assumed in a parenthetical remark above!). In fact, this turns out to be easy, based on what we already have. From the two equations

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos(-x) + i\sin(-x) = \cos x - i \sin x \end{aligned}$$

we easily derive the very useful formulas

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned}$$

These formulas must continue to be valid even when used for complex values of the variable. In particular, we get

$$\begin{aligned} \sin ix &= \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^{x}}{2i} = \frac{i(e^{x} - e^{-x})}{2} \qquad \left(\text{since } \frac{1}{i} = -i\right) \\ \cos ix &= \frac{e^{i^2x} + e^{i^2x}}{2} = \frac{e^{-x} + e^{x}}{2} \end{aligned}$$

(Those who are comfortable with the hyperbolic sine and cosine functions can easily see, for what it's worth, that $\sin ix = \dfrac{-\sinh x}{i} = i \sinh x$ and $\cos ix = \cosh x$. This seems to show that those strange hyperbolic functions actually do have some intrinsic significance after all, and are not just the artificial contrivances they at first seem to be.)

From these formulas, one can see that once one allows complex values for the variable, one no longer needs three separate functions $\sin x$, $\cos x$ and e^x. The exponential function subsumes all three. This is certainly one strong argument for the use of complex numbers in calculus.

Generating Functions

When we use the power series method to solve an ordinary differential equation, we find the coefficients of the resulting series by solving for them *recursively*: We find a_1 by using the value of a_0, and a_2 by using the value of a_1, etc; or in some cases we find a_2 by using the values of both a_1 and a_0, and then solve for a_3 by using a_2 and a_1, etc.

This process can be turned on its head. If we start with a sequence of numbers $a_0, a_1, a_2, \ldots$ defined recursively, we can consider these as the coefficients of a power series. This power series is called the **generating function** for the sequence in question (or for the recursion relation that determines it), and can sometimes provide considerable insight into the sequence.

One of the most famous recursive sequences is the **Fibonacci numbers:** 1, 1, 2, 3, 5, 8, 13, 21, ... This sequence, which arises in many different parts of nature, is characterized by the fact that each number is the sum of the preceding two. It is natural to ask if there is an algebraic formula that enables one to calculus the n^{th} element in this sequence without needing to calculate all the preceding $n-1$. The method of generating functions is an ingenious way of finding such a formula.

If we write a_i for the i^{th} number of this sequence, beginning with a_1, and in addition set $a_0 = 0$, then we have

$$a_i = a_{i-1} + a_{i-2}, \qquad a_0 = 0,\ a_1 = 1\,.$$

Now let $f(x)$ be the power series with a_i as the coefficients. Then we have

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots \\
xf(x) &= a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots + a_{n-1}x^n + \cdots \\
x^2f(x) &= a_0x^2 + a_1x^3 + a_2x^4 + \cdots + a_{n-2}x^n + \cdots\,.
\end{aligned}$$

If we now use the known fact that $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$, we see that

$$\begin{aligned}
xf(x) + x^2f(x) &= (a_1 + a_0)x^2 + \cdots + (a_{n-1} + a_{n-2})x^n + \cdots \\
&= a_2x^2 + \cdots + a_nx^n + \cdots \\
&= f(x) - a_1x - a_0 = f(x) - x
\end{aligned}$$

From this we can solve to get

$$(1 - x - x^2)f(x) = x$$

and so, completing the square in the denominator and using partial fractions,

$$\begin{aligned} f(x) &= \frac{x}{1-x-x^2} \\ &= \frac{x}{(1-\frac{x}{2})^2-\frac{5}{4}x^2} \\ &= \frac{1/\sqrt{5}}{1-\left(\frac{1+\sqrt{5}}{2}\right)x} \quad - \quad \frac{1/\sqrt{5}}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}. \end{aligned}$$

But each of these ugly looking fractions can be expanded as a geometric series, so we get

$$\begin{aligned} f(x) =& \frac{1}{\sqrt{5}}\left(1+\left(\frac{1+\sqrt{5}}{2}\right)x+\left(\frac{1+\sqrt{5}}{2}\right)^2x^2+\left(\frac{1+\sqrt{5}}{2}\right)^3x^3+\cdots\right) \\ &-\frac{1}{\sqrt{5}}\left(1+\left(\frac{1-\sqrt{5}}{2}\right)x+\left(\frac{1-\sqrt{5}}{2}\right)^2x^2+\left(\frac{1-\sqrt{5}}{2}\right)^3x^3+\cdots\right) \end{aligned}$$

Since by assumption the coefficients of the power series for $f(x)$ are the Fibonacci numbers, we thus see that

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

(It is quite remarkable that this formula even produces values which are integers, much less that it produces the Fibonacci numbers. I think that students should be encouraged to be skeptical and actually try this formula out for a few values of n, using a calculator.)