Some Comments on the Derivative of a Vector with applications to angular momentum and curvature

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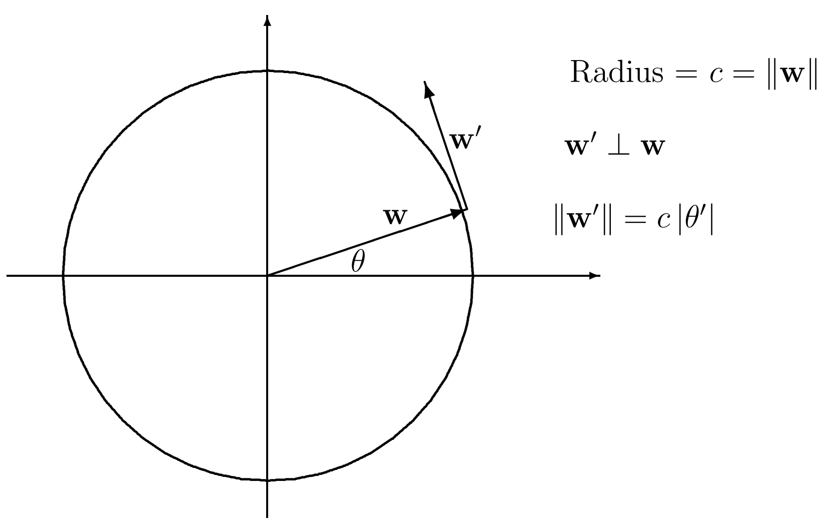
Finding the formula in polar coordinates for the angular momentum of a moving particle around the origin (the essential step in proving Kepler's Second Law of planetary motion) and computing the curvature of a parametrized curve in space can both be done by taking the cross product of a vector with its derivative. In the first case, the vector is the position vector for the moving particle. In the second, it is the velocity vector for the curve. The basic reasoning is the same in both cases, but because the symbols involved are very different, this is not apparent on first glance.

Lemma. If $\mathbf{w}(t)$ is a vector with constant magnitude, then $\mathbf{w}'$ is orthogonal to $\mathbf{w}$ and the magnitude of $\mathbf{w}'$ equals $\|\mathbf{w}\|$ times the rate at which $\mathbf{w}$ is turning (measured in radians per unit time).

PROOF: It is enlightening to first consider a proof in the two dimensional case. If $\mathbf{w}(t)$ is a plane vector with constant magnitude c, then when we position $\mathbf{w}(t)$ at the origin, its tip moves along a circle of radius c with a speed of $c\,|\theta'(t)|$ (where θ is the usual polar coordinates parameter).



Then $\mathbf{w}'$ is the velocity vector for this motion, thus

$$\|\mathbf{w}'(t)\| = c\,|\theta'|\,,$$

which is in fact $||\mathbf{w}||$ times the rate at which $\mathbf{w}$ is turning. Furthermore, $\mathbf{w}'$ is in this case tangent to the circle, and thus by elementary geometry is perpendicular to the radius vector, which is $\mathbf{w}$.

For a more computational proof in the two-dimensional case, note that by elementary trigonometry

$$\mathbf{w} = \|\mathbf{w}\|\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,,$$

so that if $\|\mathbf{w}\|$ is constant,

$$\mathbf{w}' = \theta'\,\|\mathbf{w}\|\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\,.$$

Since the factor in parenthesis has magnitude 1 and is perpendicular to $\mathbf{w}$ (see the previous formula), the result follows.

For the three-dimensional (or higher dimensional) case, start with the fact that

$$\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 = \text{constant}\,.$$

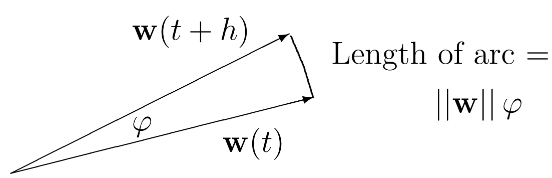
Differentiating this yields,

$$2\mathbf{w}\cdot\mathbf{w}' = \mathbf{0}\,,$$

showing that $\mathbf{w}'$ is orthogonal to $\mathbf{w}$. Now

$$\mathbf{w}'(t) = \lim_{h\to 0}\frac{\mathbf{w}(t+h)-\mathbf{w}(t)}{h}\,.$$

Visually, the numerator of this fraction looks like this:



If φ is the angle between $\mathbf{w}(t+h)$ and $\mathbf{w}(t)$, then the rate at which $\mathbf{w}(t)$ is turning is $\lim_{h\to 0+}\dfrac{\varphi}{h}$. (We assume that φ is by definition considered positive — in three dimensions, the concept of a negative angle doesn't make sense — so in taking the limit we should only consider positive values of h or, equivalently, replace h by $|h|$ in the denominator.) By assumption, $\|\mathbf{w}(t+h)\| = \|\mathbf{w}(t)\|$. For h small, φ is small and

$\|\mathbf{w}(t+h) = \mathbf{w}(t)\|$ can be closely approximated by the length $\|\mathbf{w}\|\varphi$ of the arc of a circle of radius $\|\mathbf{w}\|$ and central angle φ (see the picture). I. e.

$$\|\mathbf{w}'(t)\| = \lim_{h\to 0+} \frac{\|\mathbf{w}(t+h) - \mathbf{w}(t)\|}{h} = \lim_{h\to 0+} \frac{\|\mathbf{w}\|\,\varphi}{h} = \|\mathbf{w}\| \lim_{h\to 0+} \frac{\varphi}{h}\,,$$

and this equals $\|\mathbf{w}\|$ times the rate at which $\mathbf{w}$ is turning. (For a more careful argument, one can use the Law of Cosines to see that
$\|\mathbf{w}(t+h) - \mathbf{w}(t)\|^2 = \|\mathbf{w}\|\,(2 - 2\cos\varphi)$ and then prove that

$$\lim_{\varphi\to 0} \frac{\sqrt{2 - 2\cos\varphi}}{\varphi} = 1\,.) \quad \boxed{\checkmark}$$

Proposition A. If $\mathbf{v}(t)$ is a vector function of time, then $\dfrac{d\mathbf{v}}{dt}$ can be written as the sum of two orthogonal components,

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}')_{\|} + (\mathbf{v}')_{\perp}\,,$$

the first one parallel to $\mathbf{v}$ and the second one orthogonal to $\mathbf{v}$, with

$$\|\,(\mathbf{v}')_{\|}\,\| = \left|\,\frac{d\,\|\mathbf{v}\|}{dt}\,\right|$$

$$\|\,(\mathbf{v}')_{\perp}\,\| = \|\mathbf{v}\| \cdot (\text{rate at which } \mathbf{v} \text{ is turning})\,.$$

Furthermore,

$$\left\|\,\mathbf{v} \times \frac{d\mathbf{v}}{dt}\,\right\| = \|\mathbf{v}\|^2 \cdot (\text{rate at which } \mathbf{v} \text{ is turning})\,.$$

PROOF: Let $\mathbf{u}(t) = \mathbf{v}(t)/\,\|\mathbf{v}(t)\|$. The $\mathbf{u}$ is a unit vector with the same direction as $\mathbf{v}$ and $\mathbf{v}(t) = \|\mathbf{v}\|\mathbf{u}$. Differentiating this equation and applying the product rule yields

$$\mathbf{v}' = \|\mathbf{v}\|'\,\mathbf{u} + \|\mathbf{v}\|\,\mathbf{u}'\,.$$

Write

$$\mathbf{v}'_{\|} = \|\mathbf{v}\|'\,\mathbf{u} \qquad \text{and} \qquad \mathbf{v}'_{\perp} = \|\mathbf{v}\|\,\mathbf{u}'\,.$$

Clearly $\mathbf{v}'_{\|}$ is parallel to $\mathbf{v}$ and $\left\|\,\mathbf{v}'_{\|}\,\right\| = \left|\dfrac{d\,\|\mathbf{v}\|}{dt}\right|$. Furthermore, by the Lemma $\mathbf{u}'$ is perpendicular to $\mathbf{u}$ and $\|\mathbf{u}'\|$ equals the rate at which $\mathbf{u}$ is turning, which is also, of course, the rate at which $\mathbf{v}$ is turning. Therefore $\|\mathbf{v}'_{\perp}\|$ equals $\|\mathbf{v}\|$ times the rate at which $\mathbf{v}$ is turning.

Finally, notice that since $\mathbf{v} \times \mathbf{v}'_{\parallel} = \mathbf{0}$ and $\mathbf{v} \perp \mathbf{v}'_{\perp}$, so that

$$\begin{aligned} \|\mathbf{v} \times \mathbf{v}'\| &= \|\mathbf{v} \times \mathbf{v}'_{\perp}\| = \|\mathbf{v}\| \cdot \|\mathbf{v}'_{\perp}\| \\ &= \|\mathbf{v}\|^2 \cdot (\text{rate at which } \mathbf{v} \text{ is turning}). \quad \boxed{\checkmark} \end{aligned}$$

When we apply this to the position vector $\mathbf{r}(t)$ of a point in the plane, this Proposition gives us the formula in polar coordinates for the angular momentum of a moving object around the origin.

Proposition. If $r(t)$ is distance from the origin of a particle of mass m and ω is its angular speed about the origin, then the angular momentum of the particle about the origin is $mr^2\omega$. In particular, if the particle is moving in the plane and if (r, θ) are its polar coordinates, then the angular momentum about the origin is given by $mr^2 |\theta'|$.

PROOF: Applying Proposition A to the position vector $\mathbf{r}(t)$ of the particle, we get that

$$\frac{d\mathbf{r}}{dt} = (\mathbf{r}')_{\parallel} + (\mathbf{r}')_{\perp},$$

with

$$\|(\mathbf{r}')_{\perp}\| = \|\mathbf{r}\| \cdot (\text{the angular speed of the particle}) = \|\mathbf{r}\| \cdot \omega .$$

Here $\mathbf{r}'$ is of course the velocity of the particle and the angular momentum is given by either $mr \|(\mathbf{r}')_{\perp}\|$ or, equivalently, $m \|\mathbf{r} \times \mathbf{r}'\|$. Thus by Proposition A the angular momentum equals

$$m \|\mathbf{r}\|^2 \omega = mr^2\omega .$$

If the particle is moving in the plane, then the r here is simply the usual r for polar coordinates and $\omega = |d\theta \,/\, dt|$. $\boxed{\checkmark}$

If we now apply Proposition A to the velocity vector of a moving point, then we can derive the formula for the curvature of a parametrized curve in $\mathbb{R}^3$. Recall that curvature $\kappa(t)$ is defined to be the rate at which the unit tangent vector is turning when one moves along a curve at a speed of one unit per second. It follows that when one moves according to the given parametrization, then the rate at which the velocity vector $\mathbf{v}(t)$ is turning equals $\kappa(t) \|\mathbf{v}\|$, where $\|\mathbf{v}\|$ is of course speed.

Proposition. Let $\mathbf{r}(t)$ be the parametrization of a curve in $\mathbb{R}^3$ and $\mathbf{v}(t)$ the velocity vector and $\mathbf{a}(t)$ the acceleration vector and $\kappa(t)$ the curvature. Let $s(t)$ denote arc length (i. e. distance traveled along the curve), $\mathbf{T}(t) = \mathbf{v}/\|\mathbf{v}\|$ the unit tangent vector, and $\mathbf{N}$ the principal normal vector for the curve. Then

$$\mathbf{a} = \frac{d^2s}{dt^2}\,\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\,\mathbf{N}$$

and

$$\kappa(t) = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}\,.$$

PROOF: Since $s(t)$ is arc length, $ds/dt = \|\mathbf{v}\|$. According to Proposition A,

$$\mathbf{a} = \mathbf{v}' = \mathbf{a}_{\|} + \mathbf{a}_{\perp}$$

where since $\kappa\,\|\mathbf{v}\|$ is the rate at which $\mathbf{v}$ is turning, the directions and magnitudes are such that

$$\mathbf{a}_{\|} = \frac{d\,\|\mathbf{v}\|}{dt}\,\mathbf{T} = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}$$

$$\begin{aligned}\mathbf{a}_{\perp} &= \|\mathbf{v}\|\cdot\kappa\,\|\mathbf{v}\|\,\mathbf{N}\\ &= \|\mathbf{v}\|^2\kappa\,\mathbf{N}\\ &= \left(\frac{ds}{dt}\right)^2\kappa\,\mathbf{N}\,.\end{aligned}$$

Furthermore, according to the Proposition A,

$$\begin{aligned}\|\,\mathbf{v}\times\mathbf{a}\,\| &= \left\|\,\mathbf{v}\times\frac{d\mathbf{v}}{dt}\,\right\| = \|\mathbf{v}\|^2\cdot(\text{rate at which }\mathbf{v}\text{ is turning})\\ &= \|\mathbf{v}\|^2\cdot\kappa(t)\,\|\mathbf{v}\|\\ &= \|\mathbf{v}\|^3\,\kappa\end{aligned}$$

so that

$$\kappa = \frac{||\mathbf{v}\times\mathbf{a}||}{\|\mathbf{v}\|^3}\,. \quad \boxed{\checkmark}$$