MATH 205
APPLICATIONS OF INTEGRATION

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We will show how to derive formulas $Q = \int_a^b g(x)\,dx$, where Q is a given variable depending on a continuous function $g(x)$ defined between $x = a$ and $x = b$. We assume that Q is increasing with respect to the function g, i.e. making g bigger always makes Q bigger. We will also assume that $Q = 0$ whenever $a = b$.

Basically what will be seen is that if a formula of this sort gives the right answer for a particular variable Q whenever $g(x)$ is a constant function, then it will work for all continuous functions $g(x)$.

Hopefully, the examples that follow will make clear what is meant by a variable Q which depends on a function g.

Examples.

(1) Q is the area under the graph $y = g(x)$ between the endpoints a and b.

(2) Q is the distance traveled between time a and time b by an object moving at (variable) speed $g(x)$.

(3) Q is the volume of the solid obtained by revolving around the x-axis the graph of the function $f(x)$ between the endpoints a and b. In this case, we will choose $g(x) = \pi f(x)^2$.

(4) Q is the volume of the solid obtained by revolving around the y-axis the graph of the function $f(x)$ between the endpoints a and b. In this case, we will choose $g(x) = 2\pi x f(x)$.

(5) Q is the work done by a force $F = g(x)$ acting on an object moving between points $x = a$ and $x = b$.

Solids of Revolution. The canonical application of integration is the problem of finding the volume of the solid obtained by revolving the area under the graph of a function $y = g(x)$ around the x-axis or y-axis.

Our point of view is that a solid of revolution is simply a misshapen cylinder. It is well known that the volume of a cylinder is given by the formula $V = \pi R^2 H$, where R is the radius and H the height.

If the cylinder is positioned **horizontally**, so that the x-axis becomes its axis, with the base at position $x = a$ and the other end at $x = b$, then this formula can be written as

$$V = \pi R^2 H = \pi R^2 (b - a) = \pi \int_a^b R^2 \, dx.$$

Now if we now allow the radius R to become a variable quantity $g(x)$ as the variable x moves through the cylinder, then we now have a solid of revolution and the formula for the volume should be modified to read

$$V = \pi \int_a^b g(x)^2 \, dx.$$

If a cylinder with height H is positioned **vertically**, so that the y-axis becomes its axis, and if the radius is given by $R = b$, then the volume can be given by the formula

$$V = \pi R^2 H = \pi b^2 H = \pi \int_0^{b^2} H \, d(x^2) = 2\pi \int_0^b xH \, dx.$$

(The first integral looks slightly strange, but we can think of it as simply a shorthand for a change of variables $u = x^2$. In other words, for practical purposes, $d(x^2) = 2x \, dx$.) If we now allow the height H to become a variable quantity $h(x)$ as the variable x moves from the center of the cylinder outwards, then the formula for volume should be modified to read

$$V = \pi \int_0^{b^2} h(x) \, d(x^2) = 2\pi \int_0^b xh(x) \, dx.$$

There is a charm to this way of deriving the standard formulas for the volume of a solid of revolution, but at first the thinking behind it seems a little dubious. However it can in fact be justified.

THEOREM. Suppose that Q is a quantity that depends on a function $g(x)$ defined between a and b. (In formal notation, we can write $Q = Q(g, a, b)$.) Suppose further that Q is increasing as a function of g (i. e. making the function g larger always makes Q larger) and is additive over disjoint intervals, and that whenever $g(x) = m$, where m is a constant, then for all values a and b, $Q = (b - a)m$. Then it will be true that for every specific choice of a continuous function g,

$$Q = \int_a^b g(x)\,dx.$$

PROOF: Consider any given function $g(x)$. Hold a fixed and write $Q(x)$ for the value Q takes when we consider the function between a and x instead of between a and b. Since $Q(a) = 0$, by the Fundamental Theorem of Calculus

$$Q(b) = Q(b) - Q(a) = \int_a^b Q'(x)\,dx.$$

Therefore it suffices to prove that $Q'(x) = g(x)$. Now

$$Q'(x) = \lim_{h \to 0} \frac{Q(x+h) - Q(x)}{h}.$$

If g were a constant function g(x)=m, then the basic assumption about Q could be applied to the interval with endpoints x and $x + h$ [or the interval from $x - h$ to x in case h is negative] to show that $Q(x + h) - Q(x) = mh$ in this special case. For a non-constant function $g(x)$, if m is the minimum value that g takes between x and $x + h$ and M is the maximum, then $m \le g(x') \le M$ for all x' between x and $x + h$ and so applying the constant function case to the constants m and M yields $mh \le Q(x + h) - Q(x) \le Mh$ (because of the assumption that Q increases when the function gets larger), and so

$$m \le \frac{Q(x+h) - Q(x)}{h} \le M.$$

Now m and M actually depend on h, and since g is continuous they both converge to $g(x)$ when h approaches 0 (with x being held constant):

$$\lim_{h \to 0} m(h) = \lim_{h \to 0} M(h) = g(x).$$

Thus by the Pinching Theorem,

$$g(x) = \lim_{h \to 0} m(h) \le \frac{Q(x+h) - Q(x)}{h} = Q'(x) \le \lim_{h \to 0} M(h) = g(x).$$

Therefore $Q'(x) = g(x)$ and so $Q(b) = \displaystyle\int_a^b Q'(x)\,dx = \int_a^b g(x)\,dx$. $\boxed{\checkmark}$

Application to examples. (1) Since the area under the graph of a horizontal line $g(x) = m$ between $x = a$ and $x = b$ is just $(b - a)m$, the theorem shows that the area under the graph of any function $g(x)$ between the endpoints $x = a$ and $x = b$ is $\int_a^b g(x)\,dx$.

(2) Since the distance traveled between time $x = a$ and $x = b$ by an object moving at a constant speed m is $(b - a)m$, the theorem shows that the distance traveled between time $x = a$ and time $x = b$ by an object moving at a variable speed $g(x)$ is $\int_a^b g(x)\,dx$.

(3) If graph of a constant function $f(x) = m$ between $x = a$ and $x = b$ is revolved around the x-axis, the solid obtained is a horizontal cylinder with radius m and length $b - a$, so the volume is $\pi(b - a)m^2$. Thus the theorem shows that the volume of the solid obtained by revolving around the x-axis the graph of the function $f(x)$ between the endpoints a and b is $\pi\int_a^b f(x)^2\,dx$.

(4) If the graph of a constant function $f(x) = m$ between $x = a$ and $x = b$ is revolved around the y-axis, the solid obtained is a vertical cylindrical shell with inner radius a and outer radius b. Its volume is $(\pi b^2 - \pi a^2)m$, which can be also written as $2\pi\int_a^b mx\,dx$. Although this doesn't quite fit the pattern of the theorem, the same logic shows that the volume of the solid obtained by revolving around the y-axis the graph of any function $f(x)$ between the endpoints a and b is $2\pi\int_a^b xf(x)\,dx$.

(5) The work done by a constant force $g(x) = m$ acting on an object moving between $x = a$ and $x = b$ is $(b - a)m$. Thus the theorem shows that the work done by a variable force $g(x)$ acting on an object moving from $x = a$ to $x = b$ is $\int_a^b g(x)\,dx$.