

CURVATURE

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The curvature of a curve is, roughly speaking, the rate at which that curve is turning. Since the tangent line or the velocity vector shows the direction of the curve, this means that the curvature is, roughly, the rate at which the tangent line or velocity vector is turning.

There are two refinements needed for this definition. First, the rate at which the tangent line of a curve is turning will depend on how fast one is moving along the curve. But curvature should be a geometric property of the curve and not be changed by the way one moves along it. Thus we define curvature to be the absolute value of the rate at which the tangent line is turning when one moves along the curve at a speed of one unit per second.

At first, remembering the determination in Calculus I of whether a curve is curving upwards or downwards (“concave up or concave down”) it may seem that curvature should be a signed quantity. However a little thought shows that this would be undesirable. If one looks at a circle, for instance, the top is concave down and the bottom is concave up, but clearly one wants the curvature of a circle to be positive all the way round. Negative curvature simply doesn’t make sense for curves.

The second problem with defining curvature to be the rate at which the tangent line is turning is that one has to figure out what this means.

The Curvature of a Graph in the Plane.

In the plane, the situation is clear. If φ is the angle between the tangent line and the x -axis, then one defines the curvature to be

$$\kappa = \left| \frac{d\varphi}{ds} \right|,$$

where s is arc length. (I. e. s measures distance as one travels along the curve.)

By the chain rule, $\frac{d\varphi}{ds} \frac{ds}{dx} = \frac{d\varphi}{dx}$. Now write $\nu = \frac{ds}{dx}$. Then

$$\left| \frac{d\varphi}{ds} \right| = \left| \frac{d\varphi}{dx} \right| / \nu.$$

Thus to find the curvature, it suffices to find $\frac{d\varphi}{dx}$ and to find ν .

Clearly $s = \int \nu dx$. If one remembers the formula for arc-length, one can then anticipate that

$$\nu = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(Below, we will derive this formula.) The intuitive significance of ν is that it is the **speed** at which a point travels along the curve when its x -coordinate increases at a rate of one unit/second. (Thus the formula $s = \int \nu dx$ says that in order to compute distance, one integrates speed.)

Since φ is the angle between the direction in which the point on the curve is moving and the direction of the x -axis (i. e. horizontal), one can see that $\nu = \sec \varphi$. Since $\tan \varphi$ is the slope of the curve, i. e. $\tan \varphi = dy/dx$, we get

$$\nu^2 = \sec^2 \varphi = 1 + \tan^2 \varphi = 1 + \left(\frac{dy}{dx}\right)^2,$$

which is essentially the formula for ν anticipated above.

(An alternative explanation is to derive the formula $\nu = \sec \varphi$ by starting with the standard formula for arc length,

$$s = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

to see that

$$\nu = \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 \varphi} = \sec \varphi.$$

(Note that φ is by definition an acute angle, so $\sec \varphi \geq 0$.)

The formula for the curvature of the graph of a function in the plane is now easy to obtain. Since φ is the angle of the tangent line, one knows that $\tan \varphi$ is the slope the curve at a given point, i. e.

$$\tan \varphi(x) = \frac{dy}{dx}.$$

Differentiating with respect to x yields (by the chain rule)

$$\sec^2 \varphi \frac{d\varphi}{dx} = \frac{d^2y}{dx^2},$$

and so

$$\frac{d\varphi}{dx} = \frac{y''}{\sec^2 \varphi} = \frac{y''}{\nu^2} = \frac{y''}{1 + \left(\frac{dy}{dx}\right)^2},$$

and so

$$\kappa = \left| \frac{d\varphi}{ds} \right| = \frac{|y''|}{\nu^2} \frac{1}{\nu} = \frac{|y''|}{\nu^3} = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

Curves in Parametric Form in the Plane. The formula for the curvature of a curve in the plane described parametrically can easily be derived from the case just considered. But it is more enlightening to start from scratch, since the principles thus derived can then be adapted to the case of curves in three-space.

Given a curve $\mathbf{r}(t)$, we will write as usual $\mathbf{v}(t) = \mathbf{r}'(t)$. If we think of $\mathbf{r}(t)$ as being a moving point, then \mathbf{v} is the velocity vector, and the direction of the curve is the same as the direction of \mathbf{v} .

We let $\nu(t) = \|\mathbf{v}(t)\|$ and write $\mathbf{T}(t) = \mathbf{v}(t)/\nu(t)$. Thus $\mathbf{T}(t)$ is a unit vector with the same direction as \mathbf{v} , and is usually called the **unit tangent vector**. The calculation of the curvature depends on the following fact:

Theorem. If φ is the angle between \mathbf{v} and the positive real axis, then $d\mathbf{T}/dt$ is orthogonal to \mathbf{T} and

$$\left\| \frac{d\mathbf{T}}{dt} \right\| = \left| \frac{d\varphi}{dt} \right|.$$

Therefore if $\kappa(t)$ is the curvature, then

$$\kappa(t) = \frac{1}{\nu} \left\| \frac{d\mathbf{T}}{dt} \right\|.$$

PROOF: If we move $\mathbf{T}(t)$ to the origin, then since it is a unit vector, it becomes the radius vector for a point moving in a circle with radius 1. $\frac{d\mathbf{T}}{dt}$ is the the velocity vector for this moving point, and thus is tangent to that circle, hence is perpendicular to \mathbf{T} . Furthermore, $\left\| \frac{d\mathbf{T}}{dt} \right\|$ is the speed at which \mathbf{T} moves around that circle. Since the circle has radius 1, the angle φ of \mathbf{T} is also the distance measured along the circumference, and since speed is the derivative of distance, this speed is thus $\frac{d\varphi}{dt}$.

Now almost by definition,

$$k(t) = \frac{1}{\nu} \left| \frac{d\varphi}{dt} \right| = \frac{1}{\nu} \left\| \frac{d\mathbf{T}}{dt} \right\|. \quad \square$$

SECOND PROOF For a less “conceptual” proof, notice that since $\mathbf{T}(t)$ is a unit vector with an angle of φ to the polar axis, simple trigonometry yields

$$\mathbf{T}(t) = \cos \varphi(t) \mathbf{i} + \sin \varphi(t) \mathbf{j}.$$

It follows from the chain rule that

$$\frac{d\mathbf{T}}{dt} = \varphi'(t)(-\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j}),$$

and this vector is clearly orthogonal to \mathbf{T} and has magnitude $|\varphi'(t)|$, as claimed. \square

This formula is in practice rather unwieldy. It requires one to differentiate $\mathbf{T} = \mathbf{v}/\nu$, and ν is generally given by a rather messy square root. The following example illustrates the difficulty.

Example. Consider the ellipse

$$\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}.$$

We get

$$\mathbf{v} = \mathbf{r}' = -a \sin t \mathbf{i} + b \cos t \mathbf{j},$$

and so

$$\mathbf{T} = \frac{\mathbf{v}}{\nu} = \frac{-a \sin t \mathbf{i} + b \cos t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}.$$

Calculating $d\mathbf{T}/dt$ is quite ugly, namely

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= \frac{1}{2}(a^2 \sin^2 t + b^2 \cos^2 t)^{-3/2}(2a^2 \sin t \cos t - 2b^2 \cos t \sin t)(-a \sin t \mathbf{i} + b \cos t \mathbf{j}) \\ &\quad + (a^2 \sin^2 t + b^2 \cos^2 t)^{-1/2}(-a \cos t \mathbf{i} - b \sin t \mathbf{j}) \\ &= \frac{(a^2 - b^2) \sin 2t (-a \sin t \mathbf{i} + b \cos t \mathbf{j}) - 2\nu^2 (a \cos t \mathbf{i} + b \sin t \mathbf{j})}{2\nu^3}, \end{aligned}$$

where $\nu = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$.

The thought of finding the magnitude of this vector is likely to intimidate even the most diligent adherent to the doctrine of brute force, although the fact that the two main summands in the numerator of this fraction are orthogonal to each other simplifies things somewhat. Even so, it takes considerable work to find the curvature in simplified form,

which turns out to be $k(t) = \frac{ab}{\nu(t)^3}$.

Parametric Curves in General. The purpose of deriving the above formula

$$k(t) = \frac{1}{\nu} \left\| \frac{d\mathbf{T}}{dt} \right\|$$

for a curve in the plane given parametrically is not that it is the most practical way of determining curvature in that case. In fact, we will be able to derive a much simpler formula in that case. But this formula will depend on some insights that apply just as well to the more difficult problem of determining curvature of curves in three-dimensional or even higher dimensional space.

In 3-space, we cannot define curvature as $\nu^{-1}d\varphi/dt$, because a direction in 3-space cannot be described by an angle. To define φ as the angle between the tangent line to the curve and the z -axis, for instance, or the xy -plane will not work. This is clear when one considers the fact that if φ is defined in this way, then φ is identically 0 for a horizontal circle (or indeed any horizontal curve), but clearly one does not want the curvature to be 0 in this case.

For a curve in 3-space, we still want the curvature to be equal to the rate at which the tangent to the curve turns as one moves along the curve at a speed of one, but we need to find a way of defining what this means. What is needed is to take the angle between the tangent vectors to the curve at two nearby points $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$ and then divide by the distance between these two points. The exact curvature will then be the limit of this ratio as h approaches 0.

More practically, we can write

$$k(t) = \frac{1}{\nu} \lim_{h \rightarrow 0} \frac{\alpha_t(h)}{h},$$

where $\alpha_t(h)$ is the angle between the two vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$.

Two points to note here about $\alpha_t(h)$. First of all, since vectors do not have any fixed location, it does make sense to talk about the angle between these two vectors. For practical purposes, this means that we move the vectors $\mathbf{r}'(t)$ and $\mathbf{r}'(t+h)$ to a common location and then measure the angle between them.

Secondly, we will consider this angle to always be positive. In fact, whereas in the plane we consider angles to be positive if measured clockwise and negative if counter-clockwise, in 3-space there is no consistent way of assigning a sign to an angle.

If we imagine an airplane flying along the curve, and imagine that the pilot has a way of keeping track of the direction he is flying in in space (or, more likely, this determination could be made on the ground by a control tower), then the pilot can determine his curvature by measuring the angle between his directions at two nearby points, and then dividing by the distance traveled (i. e. dividing by νh , where h is the elapsed time).

Needless to say, this “airplane method” is not very useful to a calculus student.

The key to determining the curvature of a curve given in parametric form is to notice that the acceleration vector $\mathbf{v}'(t)$ is the sum of two orthogonal components, one of which shows how fast the speed is changing, and the other shows how fast the direction of the curve is turning.

More generally, the following is true for any vector function.

Let $\mathbf{v}(t)$ be any vector function of t (time), let $\mathbf{u}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$ (the unit vector with the same direction as \mathbf{v}), and let $\mathbf{v}'(t)$ denote the derivative of \mathbf{v} with respect to t . Then \mathbf{v}' can be written as the sum of two components, one of which has the same direction as \mathbf{v} (or the opposite direction) and shows the rate at which the magnitude of \mathbf{v} is increasing or decreasing. The other component is orthogonal to \mathbf{v} and pointing in the direction that \mathbf{v} is turning, and its magnitude is the product of $\|\mathbf{v}\|$ and the rate at which \mathbf{v} is turning.

So for instance if $\mathbf{v}(2) = 5$ cm and \mathbf{v} is decreasing at a rate of .2 cm/sec and turning at a rate of $\pi/8$ radians/sec when $t = 2$, then $\mathbf{v}'(t)$ is the sum of the vector $-.2\mathbf{u}$ plus a vector in the direction towards which \mathbf{v} is turning with a magnitude of $5\pi/8$ cm/sec.

The above theorem is simply the product rule applied to the equation

$$\mathbf{v}(t) = \|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{v}(t)\| \mathbf{u}(t),$$

which yields

$$\mathbf{v}'(t) = \frac{d\|\mathbf{v}\|}{dt} \mathbf{u} + \|\mathbf{v}\| \mathbf{u}'(t).$$

(Applied to the case when \mathbf{v} is the velocity vector for a curve, this becomes the formula

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d\nu}{dt} \mathbf{T} + \nu \frac{d\mathbf{T}}{dt}$$

giving the the tangential and normal components of the acceleration vector.) Clearly $\frac{d\|\mathbf{v}\|}{dt} \mathbf{u}$ is a vector parallel to \mathbf{v} which shows how fast $\|\mathbf{v}\|$ is increasing or decreasing.

So the only thing we need to see is that the magnitude of $\mathbf{u}'(t)$ is the rate at which $\mathbf{v}(t)$ is turning, which is, of course, the same as the rate at which $\mathbf{u}(t)$ is turning.

In the plane, we derive this by noticing (as was done above for the case $\mathbf{u}(t) = \mathbf{T}(t)$) that if the unit vector $\mathbf{u}(t)$ is moved to the origin then it becomes the position vector for

a point moving in a circle with radius one. If we write, as previously, φ for the angle \mathbf{u} makes with the x -axis, then $\mathbf{u}(t) = \cos \varphi \mathbf{i} + \sin \varphi \mathbf{j}$ and it is easy to see that $\|d\mathbf{u}/dt\|$ is the speed at which this point moves around that circle, and that this is the same as $|d\varphi/dt|$.

If we look at the reasoning involved more carefully, we see that the fact that the tip of the unit tangent vector $\mathbf{u}(t)$ moves in a circle is not crucial. In general, let $\alpha_t(h)$ be the angle between $\mathbf{u}(t)$ and $\mathbf{u}(t+h)$. If we draw a circular arc from the tip of $\mathbf{u}(t)$ to $\mathbf{u}(t+h)$ (where we have located both these vectors at the origin, and also make the origin the center of this circular arc) then because the radius of this arc has length 1 (because $\mathbf{u}(t)$ is a unit vector), the length of the arc will be equal to the angle between these two vectors, i. e. to $\alpha_t(h)$. But when h is very small, so that these two vectors are very close to each other, then this arc is very close to a straight line, so that its length $\alpha_t(h)$ is almost the same as the distance between the tips of the two vectors, i. e. to $\|\mathbf{u}(t+h) - \mathbf{u}(t)\|$. Thus we get

$$\lim_{h \rightarrow 0} \frac{\alpha_t(h)}{h} = \lim_{h \rightarrow 0} \frac{\|\mathbf{u}(t+h) - \mathbf{u}(t)\|}{h} = \left\| \frac{d\mathbf{u}}{dt} \right\|,$$

which just says that $\left\| \frac{d\mathbf{u}}{dt} \right\|$ is the rate at which $\mathbf{u}(t)$ is turning (and therefore also the rate at which $\mathbf{v}(t)$ is turning, since they both have the same direction).

Returning to the case where $\mathbf{v}(t)$ is the velocity vector for a curve, the corresponding unit vector is $\mathbf{T}(t)$, and we see that $\|\mathbf{T}'(t)\|$ is the rate at which the curve is turning, which, as we have seen, is $\nu(t)k(t)$, where ν is the speed and k is the curvature. Thus we recover the formula established previously for the case of a plane curve:

$$k(t) = \frac{1}{\nu} \left\| \frac{d\mathbf{T}}{dt} \right\|.$$

As the example of the ellipse in the plane showed, it is usually not practical to compute $d\mathbf{T}/dt$ directly. A less painful approach is to use the formula derived above for the tangential and normal components of the acceleration vector. Recall that by using the product rule to differentiate the formula $\mathbf{v} = \nu\mathbf{T}$, we derived

$$\mathbf{a} = \mathbf{v}' = \nu' \mathbf{T} + \nu \mathbf{T}',$$

where $\nu' \mathbf{T}$ is parallel to \mathbf{v} , and \mathbf{T}' is orthogonal to \mathbf{T} (and thus orthogonal to \mathbf{v}). Furthermore, as just seen, $\|\mathbf{T}'\| = \nu k$, where $k(t)$ is the curvature.

Let

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\mathbf{T}'}{\nu k(t)},$$

using the formula

$$k(t) = \frac{1}{\nu} \|\mathbf{T}'\|,$$

as shown above. Then \mathbf{N} is a unit vector perpendicular to the direction of the curve at the location $\mathbf{r}(t)$. \mathbf{N} is called the **unit normal** to the curve at $\mathbf{r}(t)$.

We now have

$$\mathbf{a}(t) = \mathbf{v}'(t) = \nu'(t)\mathbf{T}(t) + \nu^2(t)k(t)\mathbf{N}.$$

This expresses the acceleration vector as the sum of a vector in the direction of the curve (the **tangential component**) with magnitude $\nu'(t)$ and a component orthogonal to the curve (the **normal component**) having magnitude $\nu^2(t)k(t)$. (The normal component is proportional to the so-called “centrifugal force” that someone riding along the curve will feel. The formula above shows that doubling one’s speed will quadruple this centrifugal force — an important consideration for riders of motorcycles.)

Now since $\mathbf{N} \cdot \mathbf{T} = 0$ and $\mathbf{N} \cdot \mathbf{N} = 1$, from the equation above we get that

$$\mathbf{N} \cdot \mathbf{a} = \nu' \mathbf{N} \cdot \mathbf{T} + \nu^2 k \mathbf{N} \cdot \mathbf{N} = \nu^2 k,$$

yielding the following formula for curvature:

$$k(t) = \frac{\mathbf{N} \cdot \mathbf{a}}{\nu^2}.$$

This finally yields a reasonable formula for a curve in the plane given parametrically.

Theorem. If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a plane curve, the the curvature is given by

$$k(t) = \frac{|y'x'' - x'y''|}{\nu^3},$$

where $\nu(t) = \sqrt{x'(t)^2 + y'(t)^2}$.

PROOF: In the plane, it is easy to find the unit normal \mathbf{N} simply from the fact that $\mathbf{N} \perp \mathbf{v}$ and $\|\mathbf{N}\| = 1$, since in the plane there are only two vectors with length 1 perpendicular to any given vector. Now $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j}$ and one sees immediately that

$$\mathbf{v} \perp y'\mathbf{i} - x'\mathbf{j}$$

and $\|y' \mathbf{i} - x' \mathbf{j}\| = \sqrt{(y')^2 + (x')^2} = \nu$. Therefore

$$\mathbf{N} = \pm \frac{y' \mathbf{i} - x' \mathbf{j}}{\nu},$$

and

$$\begin{aligned} \mathbf{N} \cdot \mathbf{a} &= \pm \frac{(y' \mathbf{i} - x' \mathbf{j}) \cdot (x'' \mathbf{i} + y'' \mathbf{j})}{\nu} \\ &= \frac{|y' x'' - x' y''|}{\nu}. \end{aligned}$$

(We have seen above that $\mathbf{N} \cdot \mathbf{a} \geq 0$, and of course $\nu \geq 0$. Therefore the absolute value.)

Finally we get

$$k(t) = \frac{\mathbf{N} \cdot \mathbf{a}}{\nu^2} = \frac{|y' x'' - x' y''|}{\nu^3}. \quad \square$$

Example. Consider again the ellipse

$$\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}.$$

We get

$$\begin{aligned} x' &= -a \sin t & y' &= b \cos t \\ x'' &= -a \cos t & y'' &= -b \sin t \end{aligned}$$

so that

$$k(t) = \frac{|y' x'' - x' y''|}{\nu^3} = \frac{|-ab \cos^2 t - ab \sin^2 t|}{\nu^3} = \frac{ab}{\nu^3}.$$

Now in 3-space, we can't find \mathbf{N} so cheaply. But in 3-space, we have the cross-product available.

Consider again the equation $\mathbf{a}(t) = \mathbf{v}'(t) = \nu'(t) \mathbf{T}(t) + \nu^2(t) k(t) \mathbf{N}$. We want a way to isolate the second summand on the right hand side. To do this, note that since \mathbf{N} and \mathbf{T} are orthogonal unit vectors, the cross product $\mathbf{N} \times \mathbf{T}$ has length 1. On the other hand, the cross product of any vector with itself is always 0. Thus we get

$$\mathbf{T} \times \mathbf{a} = \nu' \mathbf{T} \times \mathbf{T} + \nu^2 k \mathbf{T} \times \mathbf{N} = \nu^2 k \mathbf{T} \times \mathbf{N}$$

and

$$\begin{aligned} \|\mathbf{v} \times \mathbf{a}\| &= \nu \|\mathbf{T} \times \mathbf{a}\| \\ &= \nu^3 k \|\mathbf{T} \times \mathbf{N}\| = \nu^3 k. \end{aligned}$$

This yields the following theorem:

Theorem. The curvature of a curve $\mathbf{r}(t)$ in three-space is given by the formula

$$k(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\nu^3}.$$