PROOF OF THE DIVERGENCE THEOREM

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Flux

To understand the notion of flux, consider first a fluid moving upward vertically in 3-space at a speed ν (measured in, for instance, cm/sec) which is constant with respect to time ("steady state flow") and also constant with respect to position in $\mathbb{R}^3$. If Ω is a region in the xy-plane, then the flux of this fluid across Ω is given by $\nu A(\Omega)$, where $A(\Omega)$ is the area of Ω. Essentially this measures the amount of fluid that flows across Ω per unit time.

Now if the fluid velocity is not vertical, then this formula should be replaced by $\nu A(\Omega)\cos\gamma$, where γ is the angle between the direction of the fluid and the vertical. (In the extreme case, when the fluid is flowing in a horizontal direction, $\gamma = \pi/2$ and the flux is 0. Intuitively, if the fluid is flowing horizontally and Ω is contained in the xy-plane, then no fluid flows across Ω.) If we represent the fluid velocity as a vector $\mathbf{v}$, this gives $\text{Flux} = A(\Omega)\,\mathbf{v}\cdot\mathbf{k}$.

If we now assume that the flow is still constant with respect to time, but variable from point to point in $\mathbb{R}^3$, then the multiplication in this formula should be replaced by an integral:

$$\text{Flux} = \iint_\Omega \mathbf{v}(x,y,0)\cdot\mathbf{k}\,dx\,dy.$$

(The z-coordinate is 0 because Ω lies in the xy-plane.)

Now if the planar region Ω is not horizontal, then the vector $\mathbf{k}$ in this formula needs to be replaced by a unit vector perpendicular to Ω, which we will denote by $\mathbf{n}$. But furthermore, since we are no longer considering a region in the plane, we should no longer integrate over Ω with respect to x and y. For the moment, let us write

$$\text{Flux} = \iint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma,$$

where S represents the non-horizontal planar region we are considering and $d\sigma$ indicates what is called a surface integral.

We will now figure out what a surface integral is and how to compute it. Let $dx\,dy$ denote the area of a very small rectangular piece of area in the xy-plane, and let $d\sigma$ denote the area of the piece of S that lies above this tiny rectangle. (Thus $dx\,dy$ is the area of the shadow of the piece of S corresponding to $d\sigma$.)

Since S is at an angle, $dx\,dy$ will be smaller than $d\sigma$. In fact, we will have $dx\,dy = \cos\gamma\,d\sigma$, where γ is the angle at which S is tilted: γ can be measured as the angle between a normal vector $\mathbf{n}$ to S and the vertical. If we choose $\mathbf{n}$, as before, so that $||\mathbf{n}|| = 1$, then $\cos\gamma = \mathbf{n}\cdot\mathbf{k}$ and

$$dx\,dy = \mathbf{n}\cdot\mathbf{k}\,d\sigma.$$

The formula

$$\text{Flux} = \iint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma,$$

defines the upward flux across a surface S even when S is curved. The only difference that it makes for S to be curved is that $\mathbf{n}$ is no longer a constant but varies from point to point on the surface.

From a practical point of view, though, it is not yet clear how to carry out the surface integral which defines flux. In order to do this, we need to think about how the surface S is to be described.

We will assume here that S is described as the graph of a function of two variables, i. e. S is given by an equation

$$z - f(x,y) = 0.$$

This means that we can also think of S as a level surface for the function of three variables $g(x,y,z) = z - f(x,y)$. This means that a normal vector $\mathbf{N}$ to S can be obtained as

$$\mathbf{N} = \nabla g = -\frac{\partial f}{\partial x}\,\mathbf{i} - \frac{\partial f}{\partial y}\,\mathbf{j} + \mathbf{k}.$$

Then we can obtain the unit normal $\mathbf{n}$ referred to above as

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||}.$$

We should note that $\mathbf{N}\cdot\mathbf{k} = 1$ so that, using what was shown above,

$$dy\,dx = \mathbf{n}\cdot\mathbf{k}\,d\sigma = \frac{\mathbf{N}\cdot\mathbf{k}}{||\mathbf{N}||}\;d\sigma = \frac{d\sigma}{||\mathbf{N}||}.$$

Then if the fluid velocity is given as $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$, we get

$$\begin{aligned}
\text{Flux} &= \iint_S \mathbf{v} \cdot \mathbf{n}\, d\sigma \\
&= \iint_S \frac{\mathbf{v} \cdot \mathbf{N}}{||N||}\, d\sigma \\
&= \iint_\Omega \mathbf{v} \cdot \mathbf{N}\; dx\, dy \\
&= \iint_\Omega (-v_1 \frac{\partial f}{\partial x} - v_2 \frac{\partial f}{\partial y} + v_3)\, dx\, dy,
\end{aligned}$$

using the fact that $\mathbf{N} = -\dfrac{\partial f}{\partial x}\,\mathbf{i} - \dfrac{\partial f}{\partial y}\,\mathbf{j} + \mathbf{k}$ and where Ω is the shadow (the projection) of the curved region S onto the xy-plane.

The Divergence Theorem

The divergence theorem says that if S is a **closed** surface (such as a sphere or ellipsoid) and $\mathbf{n}$ is the **outward** unit normal vector, then

$$\iint_S \mathbf{v} \cdot \mathbf{n}\; d\sigma = \iiint_T \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\, dx\, dy\, dz,$$

where T is the solid enclosed by S. (To say that S is closed means roughly that S encloses a bounded connected region in $\mathbb{R}^3$. This concept is analogous to that of a simple closed curve in the plane, but is a little harder to define precisely. A paper cup is not a closed surface, but a paper cup with a lid on it is.)

A closed surface can never be described as the graph of a function $z = f(x, y)$, because on a closed surface (x, y) never uniquely determines z. In order to evaluate the LHS of the divergence theorem, we will assume that the surface S can be broken into two parts, a top and a bottom, each of which can be described as the graph of a function. On the top we have $z = f^+(x, y)$ and on the bottom $z = f^-(x, y)$. Notice also that on the top surface the **outward** normal is the same as the **upward** normal, but on the bottom these are negatives of each other.

To evaluate the LHS of the divergence theorem, we break the surface integral into two parts, one integral over the top of the surface and one over the bottom.

$$\iint_S \mathbf{v}\cdot\mathbf{n}\;d\sigma = \iint_{\text{Top}} \mathbf{v}\cdot\mathbf{n}\,d\sigma + \iint_{\text{Bottom}} \mathbf{v}\cdot\mathbf{n}\,d\sigma$$
$$= \iint_\Omega -f_x^+ v_1 - f_y^+ v_2 + v_3\,dx\,dy - \iint_\Omega -f_x^- v_1 - f_y^- v_2 + v_3\,dx\,dy,$$

where the unexpect minus sign before the last integral here arises because on the bottom surface the outward unit normal $\mathbf{n}$ points downward rather than upward. The Divergence Theorem now becomes three separate identities:

$$\iint_\Omega -f_x^+ v_1\,dx\,dy - \int_\Omega -f_x^- v_1\,dx\,dy = \iiint_T \frac{\partial v_1}{\partial x}\,dx\,dy\,dz$$
$$\iint_\Omega -f_y^+ v_2\,dx\,dy - \int_\Omega -f_y^- v_2\,dx\,dy = \iiint_T \frac{\partial v_2}{\partial y}\,dx\,dy\,dz$$
$$\iint_\Omega v_3\,dx\,dy - \int_\Omega v_3\,dx\,dy = \iiint_T \frac{\partial v_3}{\partial z}\,dx\,dy\,dz.$$

Now in order to evaluate the double integrals on the left hand sides, one must substitute $z = f^+(x,y)$ in the first integral and $z = f^-(x,y)$ in the second. (This explains why the LHS of the third equation is not 0, despite what it looks like.) Then the third equation, for instance, follows since

$$\iint_\Omega v_3(x,y,f^+(x,y)) - v_3(x,y,f^-(x,y))\;\;dx\,dy = \iint \int_{f^-(x,y)}^{f^+(x,y)} \frac{\partial v_3}{\partial z}\,dz\;\;dx\,dy$$
$$= \iiint_T \frac{\partial v_3}{\partial z}\,dz\,dx\,dy.$$

The first and second equations can be derived in similar fashion by choosing different coordinates to work with. For instance,

$$\iint_S v_1\,\mathbf{i}\cdot\mathbf{n}\,d\sigma = \iint_\Sigma v_1(g^+(y,z),y,z) - v_1(g^-(y,z),y,z)\;dy\,dz$$
$$= \iiint_T \frac{\partial v_1}{\partial x}\,dx\,dy\,dz,$$

where $x = g^-(y,z)$ describes the left-hand half of S and $x = g^+(y,z)$ describes the right-hand half and Σ is the projection of T onto the yz-plane.