

**Do any five problems.**

- (10) **1.** Compute the following Legendre symbols. **SHOW YOUR WORK!**  
a)  $\left(\frac{48}{73}\right)$       b)  $\left(\frac{135}{139}\right)$       c)  $\left(\frac{17 \cdot 23}{283}\right)$
- (10) **2.** Prove that if  $n > 2$  then  $\tau(n) < n$ .
- (15) **3.** Use Euler's Criterion to prove that if  $p$  is prime number and  $p \equiv -1 \pmod{4}$  then  $\left(\frac{-1}{p}\right) = -1$ .
- (20) **4.** Prove that if  $(a, b) = 1$  then  $a^{\varphi(b)} + b^{\varphi(a)} \equiv 1 \pmod{ab}$ .
- (20) **5.** Prove that if  $p$  is an odd prime then exactly half of the integers between 1 and  $p - 1$  are quadratic residues.
- (25) **6.** a) Prove that a positive integer  $n$  is composite if and only if  $\varphi(n) \leq n - \sqrt{n}$ .  
b) Prove that  $n$  is composite if and only if  $\sigma(n) > n + \sqrt{n}$ .
- (25) **7.** Determine what primes  $p$  satisfy the property that  $p \mid x^2 - 3$  for some integer  $x$ . Prove your answer.
- (40) **8.** a) Prove that if  $r_1, \dots, r_{\varphi(m)}$  is a reduced residue system modulo  $m$  and  $(a, m) = 1$  then  $ar_1, \dots, ar_{\varphi(m)}$  is also a reduced residue system.  
b) Prove that if  $(a, m) = 1$  then  $a^{\varphi(m)} \equiv 1 \pmod{m}$  [Euler's Theorem].  
c) Prove that for a given integer  $a$  with  $(a, m) = 1$ , if  $t$  is the smallest strictly positive integer such that  $a^t \equiv 1 \pmod{m}$ , then  $t \mid \varphi(m)$ .