MATH 420 **FINAL EXAM** Wednesday December 15, 1993

Do any seven problems.

(15) **1.** Prove that if $a \neq 1$ then $a^3 + 1$ is not prime.

(15) **2.** Prove that if $(a, b) = 1$ then $b \nmid a - b$.

(20) **3.** Prove that if $2^m - 1$ is prime then $2^{m-1}(2^m - 1)$ is a perfect number.

(20) **4.** Find the continued fraction expansions for the following real numbers:

a) $87/38$

b) $\sqrt{28}$

(30) **5.** Prove that if n is a composite number then there exists a prime divisor p of n such that $p \leq \sqrt{n}$.

(30) **6.** Use Fermat's Little Theorem to prove that if p is a prime number larger than 17 then p^{16} has the form $51k + 1$.

(30) **7.** **a)** State the Well-ordering Principle for the natural numbers.

b) Use the Well-ordering Principle and the Division Algorithm to prove that if $d = (a, b)$ then there exist integers x and y such that $ax + by = d$.

(30) **8.** Use the Well-ordering Principle to prove that the simple continued fraction expansion for any rational number is finite. (HINT: Consider the denominator.)

(35) **9.** Let q be prime number of the form $4k + 1$.

a) Explain why there exist primes p such that p does not divide $x^2 - q$ for any x.

b) Determine those primes p which divide $x^2 - 5$ for some x.

(40) **10.** **a)** Let p be an odd prime and let $a \not\equiv 0 \pmod{p}$. Prove that the congruence $x^2 \equiv a \pmod{p}$ has either no solutions or exactly two solutions c such that $0 < c < p$.

b) Prove that if p is an odd prime then exactly half the numbers between 1 and $p - 1$ are quadratic residues modulo p.