SURFACES OF REVOLUTION

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Start with a curve given parametriclly in the xz-plane, viz.

$$\beta(v) = (x(v), 0, z(v)) = (\varphi(v),\ 0,\ \psi(v)).$$

Revolving this curve around the z-axis yields a surface which we can describe in cylindrical coordinates as $r = \varphi(v)$, $z = \psi(v)$. To make things cleaner, we can let $\mathbf{R}(u)$ denote a unit vector with angle u in the xy-plane and let $\mathbf{k}$ be the usual unit vertical vector. Thus

$$\mathbf{R}(u) = (\cos u,\ \sin u,\ 0), \quad \mathbf{k} = (0, 0, 1).$$

Note that $\mathbf{R}_u(u) = (-\sin u\,, \cos u,\ 0)$ and $\mathbf{R}_u \perp \mathbf{R}$ and $\mathbf{R}_u \perp \mathbf{k}$. ($\mathbf{R}_u$ is simply the tangent vector to the unit circle in the xy-plane.) Then the surface S is parametrized by

$$\begin{aligned}\mathbf{X}(u, v) &= \varphi(v)\mathbf{R}(u) + \psi(v)\mathbf{k} \\ &= (\varphi(v)\cos u,\ \varphi(v)\sin(u),\ \psi(v)).\end{aligned}$$

Now $\mathbf{X}_v = \varphi'(v)\mathbf{R}(u) + \psi'(v)\mathbf{k}$, and so the vector $\psi'(v)\mathbf{R}(u) - \varphi'(v)\mathbf{k}$ is orthogonal to $\mathbf{X}_v$. But we also note that it is orthogonal to $\mathbf{X}_u = \varphi(v)\mathbf{R}_u$, because $\mathbf{R}_u \perp \mathbf{R}$ and $\mathbf{R}_u \perp \mathbf{k}$. Thus we get a unit normal to S by taking

$$\mathbf{N} = \frac{\psi'(v)\mathbf{R}(u) - \varphi'(v)\mathbf{k}}{\sqrt{\varphi'(v)^2 + \psi'(v)^2}}.$$

Now notice that $\mathbf{N}$ is perpendicular to $\mathbf{R}_u$, and $\mathbf{R}_u$ is independant of v. Thus $\mathbf{N}_v$ is also perpendicular to $\mathbf{R}_u$, and so of course also $\mathbf{N}_v \perp \varphi(v)\mathbf{R}_u = \mathbf{X}_u$. Of course we also know that $\mathbf{N}_v \perp \mathbf{N}$. It follows (since we know that $\mathbf{X}_u$, $\mathbf{X}_v$, and $\mathbf{N}$ are at right angles to each other) that $\mathbf{N}_v = d\mathbf{N}_p(\mathbf{X}_v)$ is in the direction of $\mathbf{X}_v$. I.e. $\mathbf{X}_v$ is an eigenvector for $d\mathbf{N}_p$. So the meridians of the surface are lines of curvature. Since $\mathbf{X}_u$ is orthogonal to $\mathbf{X}_v$, we also see that $\mathbf{X}_u$ is in the other principal direction, so the parallels are also lines of curvature. (In fact, $\mathbf{N}_u = \psi'\,\mathbf{X}_u/\varphi\sqrt{\varphi'(v)^2 + \psi'(v)^2}\,.$)

Now a parallel is a horizontal circle with radius $\varphi(v)$, and so its curvature is $1/\varphi(v)$ and its normal vector is the horizontal vector $\mathbf{n} = -\mathbf{R}(u)$. To obtain the normal curvature of S in the direction $\mathbf{X}_u$, we need to multiply $1/\varphi(v)$ by $\langle \mathbf{N}, \mathbf{n} \rangle$. This yields

$$k(\mathbf{X}_u) = \frac{-\psi'(v)}{\varphi(v)\sqrt{\varphi'(v)^2 + \psi'(v)^2}}.$$

Since $|\mathbf{X}_u|^2 = \varphi(v)^2$, this yields

$$e = \frac{-\psi'(v)\varphi(v)}{\sqrt{\varphi'(v)^2 + \psi'(v)^2}}.$$

On the other hand, we have seen that the normal vector for the meridians is the same as the normal vector $\mathbf{N}$ for the surface. And the curvature of the meridians $k(\mathbf{X}_v)$ is the same as the curvature of the original curve β, and from basic calculus this is $\dfrac{\varphi'\psi'' - \varphi''\psi'}{(\varphi'(v)^2 + \psi'(v)^2)^{\frac{3}{2}}}$. Since $|\mathbf{X}_v|^2 = \varphi'(v)^2 + \psi'(v)^2$, this gives

$$g = \frac{-\varphi'\psi'' + \varphi''\psi'}{\sqrt{\varphi'(v)^2 + \psi'(v)^2}}.$$

And finally, $f = -\langle \mathbf{N}_u, \mathbf{X}_v \rangle = 0$.