YET ANOTHER VARIATION ON THE PROOF OF THEOREM E

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We fix a point $p \in S$ and a parametrization $\mathbf{X}$ in a neighborhood of p. (We may assume without loss of generality that $p = \mathbf{X}(0,0)$.) We write $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ for the matrix of the operator $d\mathbf{N}_p$ with respect to the basis $\mathbf{X}_u$, $\mathbf{X}_v$ of $T_p(S)$. Thus $\mathbf{N}_u = a_{11}\mathbf{X}_u + a_{21}\mathbf{X}_v$, $\mathbf{N}_v = a_{12}\mathbf{X}_u + a_{22}\mathbf{X}_v$. (Note the order of the subscripts.) We know that the Second Fundamental Form is defined on a vector $\mathbf{w}$ by $II(\mathbf{w}) = -\langle \mathbf{w}, d\mathbf{N}_p(\mathbf{w})\rangle$. Translating this into coordinates yields the matrix equations

$$(1) \qquad -\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad A = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}.$$

The second matrix equation here is historically called the *equations of Weingarten.*

If we choose the parametrization $\mathbf{X}$ so that $|\mathbf{X}_u(0,0)| = 1$, $|\mathbf{X}_v(0,0)| = 1$, and $\mathbf{X}_u(0,0) \perp \mathbf{X}_v(0,0)$, then at $p = \mathbf{X}(0,0)$ we have $E = G = 1$, $F = 0$. In this case, the matrix formula (1) reduces to $A = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$. We also note that when $E(0,0) = G(0,0) = 1$, $F(0,0) = 0$, then any *unit* vector $\mathbf{w} \in T_p(S)$ will have the form $\cos\theta\, \mathbf{X}_u + \sin\theta\, \mathbf{X}_v$ for some θ.

We let $k(\mathbf{w})$ denote the (normal) curvature of S at p in the direction $\mathbf{w}$. We know that $k(\mathbf{w}) = -\langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w}\rangle \,/\, |\mathbf{w}|^2$. In particular, if $\mathbf{w}$ is a *unit* vector, then $k(\mathbf{w}) = -\langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w}\rangle$. In particular, when $E = G = 1$, $F = 0$ at p, then $e = II(\mathbf{X}_u) = k(\mathbf{X}_u)$ and $g = II(\mathbf{X}_v) = k(\mathbf{X}_v)$, and

$$k(\cos\theta\, \mathbf{X}_u + \sin\theta\, \mathbf{X}_v) = II(\cos\theta\, \mathbf{X}_u + \sin\theta\, \mathbf{X}_v) = e\cos^2\theta + 2f\sin\theta\cos\theta + g\sin^2\theta.$$

Denote the expression above by $k(\theta)$. Then the maximum and minimum curvatures occur when

$$0 = k'(\theta) = 2(g-e)\sin\theta\cos\theta + f(\cos^2\theta - \sin^2\theta) = (g-e)\sin 2\theta + 2f\cos 2\theta,$$

so that $\tan 2\theta = -2f/(g-e)$. Now notice that if θ is a solution to this, then so is $\theta + \frac{1}{2}\pi$, and there are no solutions in between these. I.e. if the curvature has an extreme value in the direction θ, then it has another extreme value in the direction $\theta + \frac{1}{2}\pi$. Since a function cannot have two maximums in a row, we realize that if the curvature is a maximum in the direction θ, then it must be a minimum in the direction $\theta + \frac{1}{2}\pi$. This shows that the principal directions are orthogonal to each other.

Still assuming $E = G = 1$, $F = 0$ at p, assume further now that $\mathbf{X}_u$ is the principal direction where the curvature is a maximum. This means that $k(\theta)$ has a maximum at $\theta = 0$, so $k'(0) = 0$ and we see that $f = 0$. We also see that since $\mathbf{X}_u \perp \mathbf{X}_v$, then $\mathbf{X}_v$ must be in the direction of minimum curvature. Putting everything together, we have in this case $e = k_1$, $g = k_2$, $f = 0$ and $A = -\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$, so that $d\mathbf{N}_p(\mathbf{X}_u) = -k_1\mathbf{X}_u$ and $d\mathbf{N}_p(\mathbf{X}_v) = -k_2\mathbf{X}_v$. Thus $-k_1$ and $-k_2$ are the eigenvectors for $d\mathbf{N}_p$. The rest of the proof of Theorem E is now routine.

Now drop all assumptions about the parametrization $\mathbf{X}$. We will see how to find the principal directions at p, i.e. the eigenvectors for $d\mathbf{N}_p$. Translating this problem into coordinates, we are looking for the eigenvectors $\begin{bmatrix} r \\ s \end{bmatrix}$ for the matrix A, i.e. for those vectors $\begin{bmatrix} r \\ s \end{bmatrix}$ such that $A\begin{bmatrix} r \\ s \end{bmatrix}$ is parallel to $\begin{bmatrix} r \\ s \end{bmatrix}$. But a vector is parallel to $\begin{bmatrix} r \\ s \end{bmatrix}$ if and only if it is perpendicular to $\begin{bmatrix} s \\ -r \end{bmatrix}$. Now using the fact that $(EG - F^2)A = -\begin{bmatrix} e & f \\ f & g \end{bmatrix}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$, we see that the principal directions at p are the directions of the vectors $r\mathbf{X}_u + s\mathbf{X}_v$ such that

$$0 = -(EG - F^2)\,[\,s \quad -r\,]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = [\,s \quad -r\,]\begin{bmatrix} e & f \\ f & g \end{bmatrix}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}.$$

With a little algebra, this equation for r and s reduces to the quadratic equation (see p. 161)

$$\begin{vmatrix} s^2 & -rs & r^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0.$$