Other Bases for Number Systems

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MATH 111 UNIT II SET 5

The reason that the Hindu-Arabic system uses 10 symbols is undoubtedly that we have 10 fingers on our two hands. We might imagine, however, an exotic Pacific island called Fivi where the inhabitants consider it blsaphermous to use the fingers of the left hand for counting. These people would then very likely develop a numbering notation based on only five symbols. After the number 4 would come the number 10_{five} (which the Fivians might call "hand.") After 14_{five} would come 20_{five} ("twohand"), and after 44_{five} would come 100_{five} (a hand of hands, which in the Fivian language might be called a "handred").

Thus the Fivians would count as follows:

$$1_{\text{five}},\ 2_{\text{five}},\ 3_{\text{five}},\ 4_{\text{five}},\ 10_{\text{five}},\ 11_{\text{five}}, \ldots,\ 43_{\text{five}},\ 44_{\text{five}},\ 100_{\text{five}}.$$

The subscript "five," of course, is being written here only for our benefit, so we don't confuse Fivian – or base five – numbers with numbers written in the normal way. We can translate from base 5 to Hindu-Arabic – i.e. "decimal," or base 10 – notation as follows:

$$3_{\text{five}} = 3, \quad 10_{\text{five}} = 5, \quad 11_{\text{five}} = 6, \quad 20_{\text{five}} = 10, \quad 100_{\text{five}} = 25,$$

$$2104_{\text{five}} = 2 \times 5^4 + 1 \times 5^3 + 0 \times 5^2 + 4 \times 5 + 3.$$

In the same way, we could write numbers in any other base. The internal circuitry of computers corresponds to a base 2 – or "binary" number system. Computer programmers often find it convenient to work in a base 16 — or "hexadeximal" — notation.

There is an obvious way of converting numbers written in other bases to decimal notation, as illustrated by the last example above. There is also an obvious though somewhat cumbersome way of converting from base 10 to other bases. The purpose of this problem set and the next one is to develop some more efficient ways of doing these conversions.

1. Consider the following pairs of base 10 numbers:
$11,\ 110 \qquad 21,\ 210 \qquad 354,\ 3540 \qquad 98754,\ 987540$.
In each case, the second number equals the first number multiplied by 10.

Now consider a similar pair of numbers in base 5.

$$2134_{\text{five}} = 2 \times 5^3 + 1 \times 5^2 + 3 \times 5 + 4$$

$$21340_{\text{five}} = 2 \times 5^4 + 1 \times 5^3 + 3 \times 5^2 + 4 \times 5.$$

Comparing these two lines, we can see that the value of a base 5 number is multiplied by 5 when you add on a 0 at the end.

Use this fact to quickly fill in the blanks below:

a) $2134_{\text{five}} = 294$, so $21340_{\text{five}} = 294 \times 5 = 1470$.

b) $11312_{\text{five}} = 832$, so $113120_{\text{five}} =$ ________.

c) $412_{\text{five}} = 107$, so $4120_{\text{five}} = 535$, $4121_{\text{five}} = 536$, $4123_{\text{five}} =$ ________.

d) $3312_{\text{five}} = 457$, and so (skipping a step) $33124_{\text{five}} =$ ________.

e) $3_{\text{five}} = 3$, $31_{\text{five}} = 16$, $314_{\text{five}} = 84$, $3142_{\text{five}} = 84 \times 5 + 2 = 422$.

2. The last calculation above shows that we can fairly quickly rewrite a base 5 number as a base 10 number by starting at the left end and working towards the right one digit at a time. Obviously the same technique could be used with any other base. For instance, you can convert 21546_8 to base 10 by filling in the blanks below:

	$2_8 = 2$
$20_8 =$ ________	$21_8 = 17$
$210_8 = 136$	$215_8 =$ ________
$2150_8 =$ ________	$2154_8 =$ ________
$21540_8 =$ ________	$21546_8 = 9062.$

Once you have practiced this method — or "algorithm" — a little, you can do it without writing so much down, as follows. (Fill in the blanks. This converts 21546_8 into base 10.)

	2	2
16	1	______
______	5	141
1128	4	1132
______	6	9062 = ANSWER.

Describe this algorithm, and use it to convert the following numbers to base 10 notation:

23313_5 $\quad 4152_6$ $\quad 7713_8$ $\quad 11010001_2$ $\quad 8152_9$ $\quad 221320_4$.

3. Now let $wxyz_5$ be an unknown four-digit number written in base 5.

(a) Think about doing a long division to divide $wxyz_5$ by 5. Let's call the resulting quotient q and the remainder r. Then we know that r is smaller than 5 and $wxyz_5 = 5q + r$. But also $wxyz_5 = wxy0_5 + z$ and z is smaller than 5 (WHY?). And the steps above show that $wxy0_5$ is a multiple of 5. From these facts, we can see that the quotient q will be equal to wxy_5 and the remainder will be z. (Using more sophisticated language, we have $wxyz_5 \equiv z \pmod 5$.)

(b) Suppose now that we want to convert 198 to base 5. We'll start by calling the answer $wxyz_5$. We are given that $wxyz_5 = 198_{10}$. Dividing 198 by 5 produces a quotient $q = 39$ and a remainder $r = 3$. Then part (a) above shows that

$$z = r = 3, \qquad wxy0_5 = 198 - 3 = 195, \qquad wxy_5 = q = 195/5 = 39.$$

(c) Now starting with the results of part (b) use the same logic to find the base 10 numbers which equal y, $wx0_5$, and wx_5. (Namely, $y = 4$, the remainder when 39 is divided by 5; $wx0_5 = 35$ and $wx_5 = 35/5 = 7$.)

(d) Similarly we find $x = 2$, $w0_5 = 5$, and $w = 1$. Putting all this together, we have discovered a base 5 representation for 198, viz. $198_{10} = 1243_5$

4. Use the same logic to write 198 in base 3, base 4, and base 9. (You can check your answers by reconverting. $198 = 3012_4$)