Divisibility Tests

E. L. Lady

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A *divisibility test* is a quick way of deciding whether one number is a multiple of another number. For instance, we know that a number is *even* (i.e. divisible by 2) if and only if the last (units) digit of the number is even. Divisibility tests are extremely useful when one is seeking the prime factorization of a number or when reducing a fraction to lowest terms.

The following fact is fundamental for finding and explaining divisibility tests:

To say that n is divisible by r is the same as saying that $n \equiv 0 \pmod{r}$.

1. We start by noticing something very simple. It should be totally obvious that 60 is a multiple of 6, that 707 is a multiple of 7, that 4808 is a multiple of 4, that 880 is a multiple of 8, that 9306 is a multiple of 3.

How do we know this?

More generally, when checking whether a number r between 0 and 9 divides a number n, we can first *reduce* any digit of n mod r. For instance, when checking for divisibility by 7 we can replace any 7 that occurs in the number by 0, replace 8 by 1, and replace 9 by 2. So if we want to see whether 7 divides 8729, we can check 1022 instead.

Explain why this works.

The following examples further illustrate this idea:

$$9958 \equiv 1150 \pmod{8}, \qquad 72 \equiv 12 \pmod{6}, \qquad 98 \equiv 10 \pmod{4}.$$

Explain how to use this trick to quickly tell whether any two digit number is a multiple of 7 or not.

2. The next category of tests is for numbers like 2, 5, 10, 4, 8, 25, 100 which are factors of 10, factors of 100, or factors of higher powers of 10. To see whether a number is divisible by one of these, it will suffice to check either the last digit of the number (units digit), or in some cases the last two or three digits.

For instance, to see whether a number is even we only need to look at the last digit. To see whether a number is a multiple of 4, we need to look at the last two digits. For instance, to test 16728, we notice that $16728 \equiv 28 \pmod 4$, so 16728 is a multiple of 4, whereas 73514 is not, since $73514 \equiv 14 \pmod 4$.

EXPLAIN!

How can you test divisibility by 5? By 10? By 25? By 8? By 100?

3. There are some bonus tricks for checking divisibility by 4 and by 8, based on the fact that 20 is a multiple of 4 and 200 is a multiple of 8. Explain the following examples:

$$74 \equiv 14 \pmod 4, \qquad 626 \equiv 26 \pmod 8, \qquad 738 \equiv 138 \pmod 8.$$

We can also remember that $100 \equiv 4 \pmod 8$. Explain the steps in the following calculation to check whether 8 divides 56732:

$$56732 \equiv 732 \equiv 132 \equiv 4 + 32 \equiv 36 \pmod 8,$$

so 56732 is *not* a multiple of 8.

Check whether the following numbers are multiples of 4:

$$128 \qquad 792 \qquad 35483 \qquad 5782 \qquad 2274 \qquad 5356.$$

Now check whether the same numbers are divisible by 8.

4. The numbers 3, 9, and 11 are not factors of any power of 10, so to test for divisibility by these one needs to look at *all* the digits of the number to be tested. (Notice that 27, for instance, is a multiple of 9 but 7 is not, and that 121 is a multiple of 11 but 21 is not.)

For 3, 9, and 11, we notice that

$$10 \equiv 1 \pmod 9, \qquad 10 \equiv 1 \pmod 3, \qquad \text{and} \qquad 10 \equiv -1 \pmod{11}.$$

Thus, for instance,

$$3457 = 3 \cdot 10^3 + 4 \cdot 10^2 + 5 \cdot 10 + 7 \equiv 3 \cdot 1^3 + 4 \cdot 1^2 + 5 \cdot 1 + 7 = 3 + 4 + 5 + 7 \pmod 3$$

and likewise $\bmod 9$. In words: A number is congruent to the sum of its digits modulo 3 or modulo 9. Therefore a number is divisible by 3 or 9 if and only if the sum of its digits is.

The rule for 11 is slightly different. Find out what it is, describe it in words, and give an example illustrating it.

Test the following numbers to see which are divisible by 3, by 9, and by 11:

$$231 \qquad 682 \qquad 495 \qquad 1276 \qquad 2214 \qquad 5346.$$