The Period for the Decimal Expansion of a Fraction

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MATH 111 UNIT IV SET 2.2

In Set 2 you learned that if the denominator of a fraction $\dfrac{a}{b}$ has at least one prime factor different from 2 and 5, then that fraction has a repeating decimal expansion and the length of the repeating pattern is *at most* $b-1$.

In Set 2.1 you learned that the decimal expansion for $\dfrac{a}{b}$ is *immediate-repeating* when b divides $10^r - 1$ for some r, and that the length of the repeating pattern is the smallest r for which this is true. In particular, this means that the length of the repeating pattern depends only on the denominator b, and not on the numerator a (always assuming that the fraction is in lowest terms).

Notice the following examples: $1/3 = .\overline{3}$; the length of the repeating pattern is 1.
$1/7 = .\overline{142857}$; the length of the repeating pattern is 6.
$1/37 = .\overline{027}$; the length of the repeating pattern is 3.

In this set we will learn more about the possibilities for the length of the repeating pattern in the decimal expansion of a fraction, especially when the denominator is a prime.

1. A rational number larger than 1 can be written in three ways: As a decimal number, such as 3.25, a fraction, such as 13/4 (sometimes called an *improper fraction* because the numerator is larger than the denominator), and a mixed number, such as $3\frac{1}{4}$.

Rewrite the following improper fractions as mixed numbers:

$$\frac{10}{7} \qquad \frac{100}{9} \qquad \frac{25}{8} \qquad \frac{1000}{13} \qquad \frac{100}{41}.$$

2. You can answer all the following questions by doing only one calculation, if you do it **by hand:**

What is the decimal expansion of $\dfrac{1}{13}$?
Divide 1000 by 13 and find the remainder.

Rewrite 1000/13 as a mixed number.

What is 10^3 on the 13-hour clock?

What rational number has the decimal expansion $76.\overline{923076}$? (Give the answer both as an improper fraction and as a mixed number.)

What rational number has the decimal expansion $.\overline{923076}$?

If you leave out the first three steps in computing $\frac{1}{13}$ (so that the first digit of the quotient is now 9), what problem does your calculation solve?

What is the sequence of remainders you get when calculating the decimal expansion of $\frac{1}{13}$? (Caution: The first remainder is 10.)

What are the powers of 10 (i.e. 10^1, 10^2, 10^3, etc.) on the 13-hour clock?

If you skip the *first* step in the calculation of $\frac{1}{13}$ (so the answer comes out $.\overline{769230}$), what fraction does your answer represent?

For what number a is $a/13 = .\overline{692307}$? $.\overline{923076}$? $.\overline{230769}$? $.\overline{307692}$?

Before you go further, be sure you understand how all these questions are related to each other. Understanding the above questions is the key to understanding what follows.

3. We can think of what might be called a circular pattern: If you start at different points in the circle and then continue clockwise, you get the decimal expansions for $\frac{1}{13}$, $\frac{10}{13}$, $\frac{9}{13}$, $\frac{12}{13}$, $\frac{3}{13}$, and $\frac{4}{13}$. These six fractions all have the same circular pattern. This does not give us the decimal expansions for $\frac{2}{13}$, $\frac{5}{13}$, $\frac{6}{13}$, $\frac{7}{13}$, $\frac{8}{13}$, or $\frac{11}{13}$. But since these fractions are in lowest terms with denominators of 13, we know that their decimal expansions

	0	
3		7
2		6
	9	

will have repeating patterns the same length as for $\frac{1}{13}$, i.e. of length 6. Do you think that these fractions will all have the same circular pattern as each other?

Why? Test your answer by finding the decimal expansions for these fractions. (HINT: Just as in question 1, a single calculation can produce all the answers.)

4. Given that $\frac{1}{41} = .\overline{02439}$. What's the quickest way to now find the decimal expansion of $\frac{2}{41}$ without a calculator. (HINT: The answer is obvious.) Now find the decimal expansions for $\frac{3}{41}$, $\frac{4}{41}$, $\frac{5}{41}$, $\frac{6}{41}$, $\frac{8}{41}$, and $\frac{11}{41}$. Altogether, you now have 8 different circular patterns. How many different fractions does each circular pattern represent. (In other words, how many different fractions have the same circular pattern as $\frac{1}{41}$, for instance? HINT: The answer is obvious.) Considering the 8 circular patterns, how many fractions does this account for altogether? Are there any more circular patterns for fractions of the form $\frac{a}{41}$? How can you be sure?

$\frac{1}{37}$ has a pattern of length 3. How many different circular patterns would you expect for fractions with 37 as denominator? (HINT: The answer is easy.) Use your calculator to find all the circular patterns.

$\frac{1}{73}$ has a pattern of length 8. How many circular patterns do you expect for fractions with 73 as denominator.

$\frac{1}{101}$ has a pattern with length 4. How many circular patterns do you expect for fractions with 101 as denominator?

CAUTION: $\frac{1}{21} = .\overline{047619}$. How many circular patterns do you expect for fractions with 21 as denominator? Use your calculator to find all these patterns. Was your expectation correct? How is 21 different from 13, 41, 37, 73, and 101?

5. EXPLAIN WHY: If b is a prime number different from 2 and 5, then the decimal expansion for $\frac{a}{b}$ must have a pattern whose length divides $b-1$.

Explain why this is not necessarily true if the denominator b is not prime.

6. Based on what you have learned, if $1/b$ has a decimal pattern of length 5, then b must divide $10^5 - 1 = 99999$. If, in addition, b is a prime number, then 5 must divide $b-1$ (WHY?), i.e. $b \equiv 1 \pmod 5$. Use these facts to find all the primes b for which the decimal expansion of $\frac{1}{b}$ has length 5.

What primes b give decimal expansions with repeating patterns of length 1? 2? 3? 4? 6? 8?