

Irrational Numbers in General  
and Square Roots in Particular

E. L. Lady

MATH 111      UNIT IV      SET 3 – ALT

In Set 2 you learned that every rational number has a decimal expansion which either terminates or repeats. Thus if there exist any decimal expansions which do not terminate and do not repeat, the numbers represented by these decimal expansions cannot be rational.

1. Consider for instance the following number:

$$r = .123456789101112131415161718192021222324252627 \dots$$

What is the pattern for the digits of this number? What will the next six digits be?

Since this number does not have a *repeating* pattern, it is not rational. In other words, if  $m$  and  $n$  are any integers, then  $\frac{m}{n}$  is never exactly equal to  $r$ . (However  $\frac{m}{n}$  might be extremely close to  $r$ . For instance  $\frac{123,456,789}{1,000,000,000}$  is very close to  $r$ . (WHY?))

Numbers which are not rational – such as  $r$  – are called **irrational** numbers.

You might wonder why anybody would need irrational numbers. Indeed, rational numbers are good enough to do a whole lot with. Since the rational numbers are closed under addition, subtraction, multiplication, and division (except for division by 0), each of the following equations has a solution which is a rational number:

$$x - \frac{2}{3} = \frac{17}{18}$$

$$x + \frac{3}{4} = \frac{1}{5}$$

$$\frac{x}{6} = \frac{5}{7}$$

$$9x = \frac{3}{16}$$

Verify the above statement by solving all of the equations given and noticing that the solutions are rational numbers.

Not all equations involving integers have solutions which are rational numbers, however. In this set we will show that if  $N$  is an integer which is not a “perfect square” (such as 9, 16, 25, 36, 49, or 64) then the solution to the equation  $x^2 = N$  is not a rational number. You are going to see that if  $a$  and  $b$  are any integers then  $\left(\frac{a}{b}\right)^2$  is never exactly equal to  $N$ , so that  $\frac{a}{b}$  is never exactly equal to  $\sqrt{N}$ .

(Actually, if  $N > 0$  then the equation  $x^2 = N$  has two solutions:  $\sqrt{N}$  and  $-\sqrt{N}$ . What we are going to show is the  $\sqrt{N}$  is always either an integer – in which case  $N$  is a perfect square – or an irrational number.)

As preparation for the proof, we will notice something about squaring fractions which are in lowest terms.

**2.** Using the fact that

$$100 = 2^2 \cdot 5^2, \quad 91 = 7 \cdot 13, \quad 247 = 13 \cdot 19, \quad 667 = 23 \cdot 29, \quad \text{and} \quad 10,127 = 13 \cdot 19 \cdot 41,$$

find the prime factorizations for  $100^2$ ,  $91^2$ ,  $247^2$ ,  $667^2$ , and  $10,127^2$ .

Based on these examples, state a general rule for the prime factorization of the square of a number.

Which of the following fractions are in lowest terms?

$$\frac{100^2}{91^2} \quad \frac{91^2}{247^2} \quad \frac{667^2}{91^2} \quad \frac{247^2}{10,127^2} \quad \frac{667^2}{10,127^2}$$

If  $a$  and  $b$  are integers, which of the following possibilities can occur?

- (1)  $\frac{a}{b}$  and  $\frac{a^2}{b^2}$  are both in lowest terms.
- (2)  $\frac{a}{b}$  is in lowest terms but  $\frac{a^2}{b^2}$  is not.
- (3)  $\frac{a}{b}$  is *not* in lowest terms but  $\frac{a^2}{b^2}$  is in lowest terms.
- (4)  $\frac{a}{b}$  and  $\frac{a^2}{b^2}$  are both not in lowest terms.

EXPLAIN WHY!

**3.** We want to show that if  $a$  and  $b$  are integers such that  $\frac{a}{b}$  is not an integer and if furthermore  $N$  is an integer, then  $\left(\frac{a}{b}\right)^2 \neq N$ . To start with, let's think about when a fraction can be an integer.

Which of the following rational numbers are integers?

$$\begin{array}{cccccc} \frac{7}{1} & \frac{12}{1} & \frac{-4}{1} & \frac{8}{2} & \frac{100}{5} & \frac{91}{13} & \frac{91}{19} \\ & & \frac{247}{7} & \frac{247}{19} & \frac{10,127}{13} & \frac{10,127}{17} & \frac{10,127}{19} & \frac{12^2}{1^2} & \frac{91^2}{13^2} \\ & & & & & \frac{247^2}{13^2} & \frac{247^2}{17^2} & \frac{247^2}{19^2} & \frac{10,127^2}{91^2} & \frac{10,127^2}{247^2} \end{array}$$

Which of these fractions is in lowest terms?

If  $a$  and  $b$  are integers, what has to be true in order for  $\frac{a}{b}$  to be in lowest terms and also to be an integer?

If  $a$  and  $b$  are integers, when is it possible for  $\frac{a}{b}$  to be in lowest terms and for  $\frac{a^2}{b^2}$  to be an integer?

If  $\frac{a}{b}$  is a fraction in lowest terms, which of the following are possible values for  $\left(\frac{a}{b}\right)^2$ ?

(✘) 4 5 6 9 14 15 16 19 24 25

26 29 34 35 36 39 44 45 46 49

How does your answer change if  $\frac{a}{b}$  is not in lowest terms?

**4.** If  $N$  is an integer and  $a$  and  $b$  are integers and  $\left(\frac{a}{b}\right)^2 = N$ , explain why  $\frac{a}{b}$  has to be an integer. Explain why  $N$  has to be a perfect square.

Explain why if  $N$  is not a perfect square then  $\sqrt{N}$  has to be irrational.

Which of the numbers listed above in (✘) have irrational square roots?