The Lazy Man Finds Some Winning Combinations

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"I guess school is a lot easier now than it used to be," said Mr. Tinker. "My boy's homework now has what they call these Diophantine equations. Things like $17x + 14y = 1$. But you only use whole numbers. When I was in school, we had to use fractions and decimals."

"You pronounce that dye-oh-FAHN-teen," the Lazy Man said. "It means an equation whose solutions are required to be integers. Actually, that makes it a lot harder. If you were allowed to use factions it would be duck soup." And he did a calculation to illustrate. (Explain!)

Mr. Tinker studied the Lazy Man's work for a while. "I see what you mean. But the thing I noticed, sometimes you can't even solve the dye oh FAHN teen equation things at all if you can't use fractions. I mean like $15x + 25y = 1$. I worked on that one a long time before I saw there wasn't any answer." (Explain!) "Finally I invented a

"Rule: You can't solve one of these dye oh FAHN teen equations anytime ____________________.

"I thought that was pretty good for just a fix-up man, except I guess it's pretty obvious."

"Almost everything in mathematics is obvious once you see it," the Lazy Man said. "It's a good rule. And you've hit on the important thing about diophantine equations. We want to be able to tell when they have solutions and when they don't. Let's take an equation like $91x + 130y = something$. Sometimes you can solve this and sometimes you can't, depending on what *something* is. So let's call a number c a **combination** of 91 and 130 if you can solve $91x + 130y = c$.

"In other words, c is a combination if you can fill in

$$c = 91\,(\quad) + 130\,(\quad).''$$

"I can understand that all right," Mr. Tinker said.

"There's nothing to understand yet," the Lazy Man said with a little irritation. "That's just a definition. Now the important thing is that combinations of combinations are still combinations of the original numbers."

"You'd better run that one by a little more slowly."

"Well, for instance 52 is a combination of 91 and 130," (the Lazy Man wrote down a calculation to show this) "and so is 39" (he wrote down another calculation). "But 13 is a combination of 52 and 39. So if we put all that together" (he wrote down two or three lines) "we see how to show that 13 is a combination of 91 and 130."

Mr. Tinker studied the Lazy Man's arithmetic for a while. "Say, wouldn't that sort of thing always work?"

"That what I've been trying to tell you. Furthermore, if you start with any two numbers, say 41 and 7, you can always find a combination that's smaller than either of them **unless**" — and the Lazy Man paused significantly — "one of them divides the other. Try it with these pairs."

(1)	80 and 19	(4 is a combination)
(2)	128 and 20	(8 is a combination)
(3)	880 and 30	(10 is a combination)

(The Lazy Man forgot to mention that we're only looking for combinations c which are strictly positive. It's too easy if we allow $c = 0$.)

Mr. Tinker frowned and did some arithmetic for a while. Finally he said, "I get it. All you have to do is ________________________________ ."

"Right. But of course that couldn't possibly work for a pair like 20, 5 or 17, 51 or 9, 81 where one of them divides the other."

Mr. Tinker tried a calculation, and then said, "Because you're not allowing 0 as an answer, right"

"Exactly," said the Lazy Man. "Now let's find the smallest possible combination of 52 and 116."

(4) $\mathbf{116} - 2 \cdot \mathbf{52} = \mathbf{12}$

(5) $\mathbf{52} - 4 \cdot \mathbf{12} = \mathbf{4}\,.$

(Numbers which are known to be combinations of 52 and 116 are darkened. This includes the original numbers themselves, since, for instance, $\mathbf{116} = 1 \cdot \mathbf{116} + 0 \cdot \mathbf{52}$.)

"How do you know there's nothing smaller than 4?" Mr. Tinker asked.

"Your RULE tells me that."

"Hey, that's right! I forgot all about my rule. My mother used to say I'd forget my head if it wasn't attached."

"Somebody should have told your mother about the subjunctive," the Lazy Man said.

1. Show why the equation $17x + 14y = 1$ can be solved extremely easily if we allow x and y to be (possibly) fractions. (There are lots of solutions. Find one.)

2. By looking at Mr. Tinker's example, find his RULE for when Diophantine equations can't be solved.

3. a) Show that 52 is a combination of 91 and 130 by solving the Diophantine equation $19x + 130y = 52$.

b) Likewise, solve the equation $91x + 130y = 39$.

c) Solve the Diophantine equation $52x + 39y = 13$.

d) Use the solutions from parts (a), (b), and (c) to solve the Diophantine equation $91x + 130y = 13$.
(Unless you use parts **(a), (b),** and **(c)** you'll miss the whole point.)
Convince yourself that this sort of thing always works, thereby proving the Lazy Man's rule "A combination of combinations is still a combination of the original numbers."

4. Use the same technique as in **3** to solve all the Diophantine equations below:

a)	b)	c)
$25x + 72y = 22$	$24x + 103y = 7$	$101x + 19y = 6$
$25x + 22y = 3$	$7x + 24y = 3$	$19x + 6y = 1$
$25x + 72y = 3$	$24x + 103y = 3$	$101x + 19y = 1$
$25x + 3y = 1$	$3x + 103y = 1$	
$25x + 72y = 1$	$24x + 103y = 1$	

5. Find a systematic way of always finding a combination of two numbers which is smaller than either of the two numbers **except** in the case where one divides the other. Illustrate by using your method on the pairs

a) 41, 7 **b)** 80, 19 **c)** 124, 20.

6. Explain why, if a divides b, then no (strictly positive) combination of a and b could possibly be smaller than a.

7. Decide what the smallest possible combination would be for each pair below:

a) 59, 30 **b)** 41, 99 **c)** 42, 105

d) 39, 93 **e)** 84, 212 **f)** 256, 162

g) 91, 105 **g)** 143, 561

If a number divides both a and b, we call that number a **common factor** of a and b. (Also known as a **common divisor** of a and b.)

8. Restate (if necessary) Mr. Tinker's RULE using the term "common factor." What does this rule say, in particular, about the relationship between the smallest combination of two numbers and their largest common factor?

9. What does the procedure you found in **5** (and the fact that a combination of combinations is still a combination of the original numbers) show about the relationship between the smallest combination of two numbers and their largest common factor?

10. Sum up by giving all possible conclusions (in addition to the ones already stated) about the relationship between the common factors of two numbers and the combinations of these same two numbers.