The Lazy Man Gets to the Root of Things
E. L. Lady

"How's the homework coming along?" the Lazy Man asked.

"I'm understanding it now," Mr. Tinker said, "but I'm not enjoying it. I've been doing square roots. They're horrible. I don't see why people have to learn this anymore anyway, since every calculator has a square root button."

"What I can't see," the Lazy Man said, "is how your learning about square roots is going to help get your son into medical school."

"Well, I figure it can't hurt. I just wish his teacher would finish these and go on to something not quite so ugly."

"Let me see you do one."

Mr. Tinker took the pencil and paper and started working. The Lazy Man looked over his shoulder. "Good grief!" he said. "No wonder you hate them so much. I'd have thought they'd have finally stopped teaching that awful method. After all, the easy way''s been around since at least 300 B. C."

"When I was in school," Mr. Tinker said, "it seemed like some of my teachers had been around that long too. How does this easy way work?"

"Suppose we want the square root of 11. Now if $x = \sqrt{11}$, then $x = \dfrac{11}{x}$."

"Why is that?"

The Lazy Man explained. "But the really useful thing is that if instead of being exactly the square root of 11, x is just a number fairly close to $\sqrt{11}$, then $\sqrt{11}$ will be close to halfway between x and $\dfrac{11}{x}$. Let's start off, for instance, with the best guess we can make easily, say $x = 3$.

Mr. Tinker, who had already punched the problem into his calculator, said, "That's not very close."

"It will do for a start. Now we want the number halfway between 3 and $11/3$. That would be

$$\frac{1}{2}\left(3 + \frac{11}{3}\right) = \frac{1}{2}\left(\frac{3^2 + 11}{3}\right) = \frac{20}{6} = \frac{10}{3}.$$

"And that's it? I think you made a mistake," Mr. Tinker said, looking at his calculator with a satisfied smile.

"Well, that's only a rough approximation. Since our first guess was not very good, we ought to repeat the process one more time. Halfway between $\dfrac{10}{3}$ and $\dfrac{11}{10/3}$ is

$$\frac{1}{2}\left(\frac{10}{3} + 11 \cdot \frac{3}{10}\right) = \frac{10^2 + 11 \cdot 9}{60} = \frac{199}{60} = 3.316666\ldots$$

Your calculator shows $\sqrt{11} = 3.3166248\ldots$, which is more accurate. But 3.31666 ought to be close enough to satisfy any high school teacher.

"Now let's try 30. Since 30 seems about half way between 25 and 36, as a first guess let's try $5\frac{1}{2} = \frac{11}{2}$.

$$\frac{1}{2}\left(x + \frac{11}{x}\right) = \frac{1}{2}\left(\frac{11}{2} + 30 \cdot \frac{2}{11}\right) = \frac{11^2 + 30 \cdot 2^2}{44} = \frac{241}{44}.$$

Now use $\dfrac{241}{44}$ as a new guess.

$$\frac{1}{2}\left(\frac{244}{44} + 30 \cdot \frac{44}{241}\right) = \frac{241^2 + 30 \cdot 44^2}{2 \cdot 44 \cdot 241} = \frac{116,162}{21,208} = 5.47727\ldots$$

Whereas in actual fact, $\sqrt{30} = 5.4772256\ldots$."

"So it's still not quite exact yet," Mr. Tinker said.

"Well, you can put it through a third time if you want still more accuracy. But it will never be quite perfect, of course, because the answers you get are fractions and these square roots can't be written as fractions."

"Is that a fact?" Mr. Tinker said. "How can you tell which square roots can be fractions and which ones have to be decimals?"

"It's easy to see why the square root of an integer can't be a fraction $\dfrac{a}{b}$. Except for the obvious cases like $\sqrt{25}$ or $\sqrt{81}$ where the square root is an integer. That's because if you take a fraction like $\dfrac{199}{60}$ with a denominator larger than 1 [assume the fraction is in lowest terms], then when you square it the answer would never be exactly an integer. For instance,

(1)
$$\left(\frac{199}{60}\right)^2 = \frac{39601}{3600} = 11\,\frac{1}{3600}$$

(2)
$$\left(\frac{577}{408}\right)^2 = \frac{332,929}{166,464} = 2\,\frac{1}{166464}.\text{"}$$

"But sometimes it might come out exactly, just by accident, wouldn't it?"

"Absolutely not. Look, if you take a fraction written in lowest terms and square it, the answer will still be in lowest terms. You see that, don't you?"

"I suppose so," aid Mr. Tinker doubtfully.

"Well, look. Take $\dfrac{21}{16}$, for instance. The only prime number dividing the denominator is 2 and it doesn't divide the numerator, so the fraction is in lowest terms. $16 = 2^4$ and $21 = 3 \cdot 7$. Now when you square $\dfrac{21}{16}$ you get $\dfrac{21^2}{16^2}$. Since $16^2 = (2^4)^2 = 2^8$ and $21^2 = (3 \cdot 7)^2 = 3^2 \cdot 7^2$, we see that 2 is still the only prime number dividing the denominator and it still doesn't divide the numerator, so the fraction is still in lowest terms.

"Or take $\dfrac{91}{10}$. Now $91 = 7 \cdot 13$ and $10 = 2 \cdot 5$. And so

$$\left(\frac{91}{10}\right)^2 = \frac{91^2}{10^2} = \frac{7^2 \cdot 13^2}{2^2 \cdot 5^2},$$

so there still aren't any common factors between the numerator and denominator, so $\dfrac{91^2}{10^2}$ is still in lowest terms."

After looking at a few more examples, Mr. Tinker said, "I see what you mean. The only primes that divide the numerator are you square it are _____ , and the same for the denominator.

"Right," the Lazy Man said. "So the answer is still in lowest terms. The square of a fraction which is not an integer can't be an integer." And he leaned back with a satisfied look.

"Now you lost me again," Mr. Tinker said. "Tell me again why the square can't be an integer."

The Lazy Man sighed. "If the answers in lowest terms, how can it reduce to an integer?"

"Oh, I see," Mr. Tinker said. "So you're saying it can't be an integer unless the denominator is 1. But if after you square it, you get a denominator which is 1, that would be okay."

The Lazy Man shook his head in exasperation. Mr. Tinker's face took on a look of concentration. Finally the tension dissolved and he smiled. "Oh, right. It would have to be an integer to start with, otherwise you couldn't get 1 in the denominator after you square it." (EXPLAIN!)   "That was pretty obvious, wasn't it? You just have to be patient with me. 'We're Tinkers, not thinkers,' my father always used to say.

"With people like your father around, it's no wonder we're a nation of mathematical illiterates," the Lazy Man said.

1.   Use the Lazy Man's technique to find good approximations to the square roots of the following numbers.

a) 10   **b)** 12   **c)** 13

d) 28   **e)** 35   **f)** 51   **g)** 75

Do not use a calculator for the arithmetic.

2.   Summarize the argument showing that $\sqrt{11}$ can't be written as a fraction as follows: Given a fraction, $\dfrac{m}{n}$. We want to show that $\left(\dfrac{m}{n}\right)^2$ can't possibly be 11.

a) We can assume that $\dfrac{m}{n}$ is in lowest terms. This means _____ .

b) No prime number can divide both m^2 and n^2 because _____ .

This means that $\dfrac{m^2}{n^2}$ is still in lowest terms.

c) Obviously the denominator of $\dfrac{m^2}{n^2}$ is not 1 because _____ .
(NOTE: **c)** is true **unless** $n = 1$. Why not choose $n = 1$?)

d) Thus we see that $\dfrac{m^2}{n^2}$ can't be 1 because _____ .

"Does your method also work for cube roots?"

"It's not my method," the Lazy Man said. "It's the method any sensible person uses. And for cube roots, use the following fact: If x is a halfway decent guess for the cube root of a number a, then the true cube root will be fairly close to one-third the way betweeen x and $\dfrac{a}{x^2}$. In other words,

$$\frac{1}{3}\left(2x + \frac{a}{x^2}\right)$$

will be a good approximation. If you repeat the process twice, your answer should be pretty accurate.

"For instance, for the cube root of 10, start with $x = 2$."

$$\frac{1}{3}\left(2 \cdot 2 + \frac{10}{2^2}\right) = \frac{1}{3}\left(\frac{2 \cdot 2^3 + 10}{4}\right) = \frac{26}{12} = \frac{13}{6}.$$

$$\frac{1}{3}\left(2 \cdot \frac{13}{6} + 10 \cdot \frac{6^2}{13^2}\right) = \frac{2 \cdot 13^2 + 10 \cdot 6^3}{3 \cdot 1014} = \frac{6554}{3042} = 2.1545036\ldots$$

In actual fact, $\sqrt[3]{10} = 2.1544347$.

"Why is the square root halfway between and the cube root one-third of the way?" Mr. Tinker asked.

The Lazy Man frowned and thought for a minute. "You don't mind if I use a little calculus, do you?"

Mr. Tinker looked alarmed. "Uh, that's all right. I don't really need to know. Besides, I promised the wife I'd be home before 8:30."