

## The Lazy Man Explains the Irrational

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“I’ve been thinking about those numbers that you can’t write as fractions,” Mr. Tinker said.

“Irrational numbers, they’re called,” the Lazy Man answered.

“Well, it all seems pretty irrational to me. I mean, what I was thinking was that when I was in school, I remember that they taught us that decimals were just another way of writing fractions. So I don’t see where these irrational numbers, like you call them, fit in. Of course I don’t suppose people knew as much back then. That was before the New Math and all.”

The Lazy Man smiled. “It’s true that, in one sense, irrational numbers have only been used since about the time of the Renaissance, which is pretty recently. (Euclid had to ban the use of numbers from geometry, because the Greeks didn’t have numbers to represent things like  $\sqrt{5}$  or  $\pi$ .) But what your teachers were talking about was decimals that end after a finite number of places, like 2.53 or .078. Non-terminating decimals may or may not be irrational.”

“Non-terminating,” Mr. Tinker said with a frown. “You mean like  $\sqrt{2}$ , which is 1.4141414141 ...?”

The Lazy man frowned and reached for his calculator. “ $\sqrt{2} = 1.4142136\dots$ ”

“Is that when it begins to repeat?” Mr. Tinker asked.

“It never does repeat,” the Lazy Man said.

“That’s the whole point. If it repeated, it would be rational instead of irrational. For instance, the number you mentioned: 1.414141 ... has to be rational. To see that, let’s call it  $R$ .

$$R = 1.41414141414141\dots$$

Now multiply it by 100.

$$100R = 141.41414141414141\dots$$

“Right,” Mr. Tinker said. “Just move the decimal point.”

“Now subtract those two.”

Mr. Tinker wrote down

$$\begin{array}{r} 100R = 141.41414141414\dots \\ -R = \underline{\underline{-1.41414141414\dots}} \end{array}$$

“I’m not sure how to do this,” he said. “Where do you start?”

“Well, certainly everything to the right of the decimal point is the same, isn’t it?” the Lazy Man said. “So you get all zeros there.” And he wrote down

$$\begin{array}{r} 100R = 141.41414141414\dots \\ -R = \underline{\underline{-1.41414141414\dots}} \\ 99R = 140.00000000000\dots \end{array}$$

<sup>2</sup> Mr. Tinker looked unhappy. “What happens when you get to the end of those decimals.? You wouldn’t still have a 0 at the end, would you?”

“But there isn’t any end,” the Lazy Man said. “They both just keep on going, so you keep getting 0’s.”

Mr. Tinker still seemed puzzled.

“I never cease to be amazed,” the Lazy Man said, “at the inability of the average person to cope with the most elementary properties of infinity. Look, it’s perfectly all right. You get  $99R = 140$  and so

$$R = \frac{140}{99}.$$

Thus  $R$  is rational — it can be written as a fraction. We can check it by working out the decimal expansion of  $140/99$ .”

$$\begin{array}{r}
 1.414\dots \\
 99 \overline{)140.000} \\
 \underline{99} \\
 **0 \\
 \underline{***} \\
 140 \\
 \underline{99} \\
 **0 \\
 \underline{***} \\
 14
 \end{array}$$

“I took it through two cycles, to make it really obvious that the pattern repeats.”

“Wait a minute. I need to fill in those asterisks.”

“No you don’t. You know that the remainder after one cycle (two steps) will be 14, because  $99 \times 140$  is 14 less than  $100 \times 140$ , which would be exactly the dividend, 14000.”

Mr. Tinker took a fresh piece of paper and did the whole calculation for itself. “Okay, I believe you.”

“Sure, it works for any repeating decimal. Take  $.723723723\dots$ , for instance. What fraction would that be?”

“I don’t have the faintest idea,” Mr. Tinker said.

“Sure you do,” said the Lazy Man. “It would be  $723/999$ . I just showed you why. If you do the division for  $723/999$ , it comes out like this:

$$\begin{array}{r}
 .723\dots \\
 999 \overline{) 723.000} \\
 \underline{722 \quad **} \\
 723
 \end{array}$$

The remainder has to be 723, because 723,000 is  $1,000 \times 723$  and  $999 \times 723$  is 723 less than that.” (EXPLAIN!) “And you can see that from that point on, it repeats.”

“What about decimals that don’t start repeating right away?” Mr. Tinker asked. “Something like 3.975757575 ... ”

The Lazy Man wrote down the following:

$$\begin{array}{r}
 100S = 397.575757575\dots \\
 -S = -3.97575757575\dots \\
 \hline
 99S = 393.6000000000\dots
 \end{array}$$

“Thus  $S = \frac{393.6}{99} = \frac{3936}{990}$  .”

Mr. Tinker studied the calculation and nodded, but still unhappily. “It just seems to me that any decimal that goes on forever would have to start repeating eventually.”

“Not at all. You could take

$$1.23456789101112131415161718192021\dots$$

Obviously that one would never repeat.”

“So it would be irrational.”

“Right.”

“Because of what you showed me.”

“Ah, no,” the Lazy Man said. “I showed you that any repeating decimal can be converted into a fraction. But now we’re talking about the converse. We need to see that if you start with a fraction and convert it to a decimal, it will always repeat?”

“You’re just saying the same thing in different words, aren’t you?”

“Not at all. What I showed you could be diagrammed like this:

$$\text{Repeating decimal} \longrightarrow \text{Fraction.}$$

But what I didn’t explain is why it would also be true that

$$\text{Fraction} \longrightarrow \text{Repeating Decimal.}$$

For instance, if I find the decimal expansion of some fraction like  $4829/762$ , how can I be sure that it will eventually start to repeat?”

“But you just showed me that it wouldn’t always have to repeat. Like that one you wrote down, .123456789101112 ... .”

“But that one didn’t come from a fraction. If you start with a fraction, you always get a decimal that repeats.”

4 “Why?”

“Well, let’s take  $\frac{2}{13}$  for instance. If we compute its decimal expansion, the beginning of the calculation looks like this:

$$\begin{array}{r} .1538 \\ 13 \overline{) 2.00000} \\ \underline{13} \phantom{00000} \\ 70 \phantom{00000} \\ \underline{65} \phantom{00000} \\ 50 \phantom{00000} \\ \underline{39} \phantom{00000} \\ 110 \phantom{00000} \\ \underline{104} \phantom{00000} \\ 60 \phantom{00000} \end{array}$$

“That’s enough to think about for a start.”

“It hasn’t started repeating yet.”

“We haven’t given it a fair chance,” the Lazy Man said. “But I want you to think about this question: Each time you get to a new step, how much of what you’ve written down before do you need to know in order to figure out what digit comes next?”

Mr. Tinker thought for a while. “All of it, I guess.”

“Not at all. Think again. What’s the next digit after what I’ve got so far?”

“4,” Mr. Tinker said.

“How much of my calculation did you look at to figure that out?”

“Well, I saw that 13 times 5 would be bigger than 60, so 13 goes into 60 4 times.”

“So you only needed to look at the bottom number: 60.”

“So what’s the point?”

“If we keep doing more steps, what are the possible numbers that might show up at the bottom of the calculation? (What are the possible remainders?)”

“I don’t know.”

“Would 10 be a possibility?”

“I don’t know. I’d have to finish the calculation.”

“Right,” the Lazy Man said. “We can’t tell whether 10 actually shows up or not without doing the whole calculation. Likewise for 20, except we know that it shows up, because that was the starting number. 30 and 40 are in doubt, but 50, 60, and 70 have already appeared. 80, 90, and 100 are in doubt, and so is 120, but we’ve already seen 110.”

“So what’s the point?”

“What about 130 or 140? Would these ever show up?”

“I can’t tell yet.”

“Sure you can. If you’re dividing by 13, the remainder you get at the bottom is never going to be 13, 14, or anything bigger.”

Mr. Tinker thought about that for a long time, and did a few experimental calculations. Finally, he said, “Oh, I get it! Of course! It’s because you’re dividing by 13.” (EXPLAIN!)

“So you can’t go through more than 12 steps without seeing the some number show up at the bottom that you’ve already encountered once before.”

“And at that point you start repeating again from the beginning,” Mr. Tinker said with a satisfied smile.

“Well, not necessarily from the beginning, but from some previous part of the calculation. For instance, look at the expansion for  $\frac{9}{74}$ .

$$\begin{array}{r}
 .1216 \\
 74 \overline{) 9.00000} \\
 \underline{74} \\
 160 \\
 \underline{148} \\
 120 \\
 \underline{740} \\
 460 \\
 \underline{444} \\
 160
 \end{array}$$

“You can see that now, after only 4 steps, it’s going to repeat, but not from the very beginning. We see that  $9/74 = .1216216216\dots$ ”

“I wonder if that’s ever going to happen to you?” Mr. Tinker asked. “That you’ll run out of new things to tell me and have to start repeating, I mean.”

The Lazy Man’s only answer was an enigmatic smile.

1. Change the following repeating decimals to fractions:

a)  $1.721721721\dots$       b)  $.97343434\dots$       c)  $14.3714371437\dots$

2. Change the following fractions to repeating decimals. Do not use a calculator.

$$\frac{1}{13} \quad \frac{9}{13} \quad \frac{12}{13} \quad \frac{3}{13} \quad \frac{1}{7} \quad \frac{2}{7} \quad \frac{3}{7} \quad \frac{1}{11} \quad \frac{1}{99}$$

3. Fill in the blanks in the Lazy Man's explanation of why the decimal corresponding to a fraction must be repeating.

4. What is the maximum number of steps that the decimal expansions for the following numbers can go before they begin to repeat?

$$\frac{1}{17} \quad \frac{1}{23} \quad \frac{1}{35} \quad \frac{1}{101} \quad \frac{1}{20} \quad \frac{1}{97} \quad \frac{1}{50}$$

5. For this problem, consider only fractions of the form  $\frac{1}{n}$  which have a repeating pattern that starts immediately after the decimal point. For instance,  $\frac{1}{37} = .027027027\dots$  would be an example, but not  $\frac{1}{15} = .06666\dots$

a) What are the numbers  $n$  such that  $\frac{1}{n}$  has a repeating pattern beginning immediately after the decimal point and consisting of a single digit. In other words,  $\frac{1}{n} = .aaaaaa\dots$

b) What are the numbers such that  $1/n$  has a pattern of length 2? I. e.  $\frac{1}{n} = .abababab\dots$

c) What numbers  $n$  give a pattern of length 3 for  $1/n$ ? Of length 4?

6. Substitute digits for the letters in the following puzzle:

$$\frac{\text{EVE}}{\text{DID}} = .\text{TALKTALKTALK} \dots$$