Basic Terminology for Systems of Equations in a Nutshell

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A system of linear equations is something like the following:

$$3x_1 - 7x_2 + 4x_3 = 10$$
$$5x_1 + 8x_2 - 12x_3 = -1.$$

Note that the number of equations is not required to be the same as the number of unknowns.

A **solution** to this system would be a set of values for x_1, x_2, and x_3 which makes the equations true. For instance, $x_1 = 3$, $x_2 = 1$, $x_3 = 2$ is a solution. We will often think of a solution as being a vector: [3, 1, 2] is a solution to the above equation. (For technical reasons, it will later be better to write solution vectors vertically rather than horizontally. For the moment, we won't worry about the way vectors are written.)

As you know from Math 231, a system of two equations can also be thought of as a single equation between two-dimensional vectors. It's easier to see things if we write these vectors vertically. The above system then becomes the vector equation

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} -7 \\ 8 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ -12 \end{bmatrix} x_3 = \begin{bmatrix} 10 \\ -1 \end{bmatrix}.$$

A system of linear equations is called **homogeneous** if the right hand side is the zero vector. For instance

$$3x_1 - 7x_2 + 4x_3 = 0$$
$$5x_1 + 8x_2 - 12x_3 = 0.$$

This system actually has a number of solutions, but there is one obvious one, namely $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This solution is called the **trivial solution**. (IMPORTANT NOTE: Trivial as used this way in Linear Algebra is a technical term which you need to know.)

Definition. A vector is called **trivial** if all its coordinates are 0, i. e. if it is the zero vector.

In Linear Algebra we are not interested in only finding one solution to a system of linear equations. We are interested in all possible solutions. In particular, **homogeneous** systems of equations (see above) are very important. The important question for a homogeneous system is whether or not there is any **non-trivial** solution, i. e. whether there is any solution other than the trivial one. Sometimes there will be, sometimes there won't. Paradoxically, it's actually the case where the trivial solution is the **only** possible one that is the most important. This situation is described by **one of the most important words in the whole course**.

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors (all the same dimension). These vectors are called **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

has **only** the trivial solution.

Example. The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are linearly independent.

How can we prove that the above is correct? We look for solutions to the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can also be written as

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 0. \end{aligned}$$

Subtracting the first equation from the second shows that $x_2 = 0$ and substitution then shows that $x_1 = 0$. Therefore the only solution is the trivial one, so the vectors are linearly independent.

Example. The vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are **not** linearly independent.

When we try high school algebra on the system

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 6x_2 &= 0 \end{aligned}$$

it doesn't work. We are unable to find a single unique solution. In fact, although $x_1 = 0$, $x_2 = 0$ is **one** solution, there is also another, **non-trivial** solution $x_1 = 3$, $x_2 = -1$. This **non-trivial** solution shows that the vectors are **not** linearly independent.

Going back to non-homogeneous systems. (I. e. the right hand side is not zero.) Unlike homogeneous systems, a non-homogeneous system might not have any solution at all. For instance, the non-homogeneous system

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

has no solution, since on the left-hand side the third coordinate of each vector is the sum of the first two, but on the right hand side this is not true, so if x_1, x_2, and x_3 satisfy the first two equations, they cannot satisfy the third. We then say that this system of equations is **inconsistent.**

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{b}$ be vectors of the same size, written vertically. The system of linear equations $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$ is called **consistent** if it has at least one solution and is called **inconsistent** if it has no solution.

If the system $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$ is consistent, we say that $\mathbf{b}$ is a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Example. The linear system

$$x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

is consistent since it has the solution $x_1 = 1,\ x_2 = 2$. Therefore $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

For a **homogeneous** system of linear equations either

(1) the system has only one solution, the trivial one;
(2) the system has more than one solution.

For a **non-homogeneous** system either

(1) the system has a single (unique) solution;
(2) the system has more than one solution;
(3) the system has no solution at all.

Which of the above possibilities occurs with a linear system has a lot to do with the coefficients on the left-hand side. We can write these coefficients in the form of a matrix, the **coefficient matrix** of the linear system.

EXAMPLE. For the non-homogeneous linear system

$$\begin{aligned} 8x_1 + 3x_2 - 5x_3 &= 12 \\ 7x_1 \phantom{{}+3x_2} + 4x_3 &= 0 \end{aligned}$$

the coefficient matrix is

$$\begin{bmatrix} 8 & 3 & -5 \\ 7 & 0 & 4 \end{bmatrix}.$$

As we go through the steps of solving a linear system by the method of **elimination**, the rows of coefficient matrix will change. These changes in the coefficient matrix are called **elementary row operations** (see book for the definition). After we do enough elementary row operations, we can eventually reduce the matrix to **row echelon form**. This means that the leading (non-zero) entry in each row is 1 except for those rows which are all 0's, and the leading entries are indented as you move down the rows.

EXAMPLE. A matrix in row echelon form:

$$\begin{bmatrix} 0 & 1 & 7 & 12 & -2 \\ 0 & 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This would mean that the original linear system has been reduced by elimination to a system looking like

$$\begin{aligned} x_2 + 7x_3 + 12x_4 - 2x_5 &= * \\ x_4 - 9x_5 &= * \\ 0 &= * \,. \end{aligned}$$

The right hand side of these equations will be derived from the right hand sides of the original system of equations.

Two important points can be noted:

• The right-hand side of the third equation in this example may turn out to be non-zero. In this case, no solution is possible and the system is therefore **inconsistent**.
• If there exists any solution at all, then x_1, x_3, and x_5 can be chosen arbitrarily; therefore there will be infinitely many solutions.

We will see that in fact these are general principles:

A system of linear equations with coefficient matrix A will be inconsistent for certain values on the right hand side if the row echelon form of A contains a row of zeros.
If the row echelon form of the coefficient matrix A does not contain a row of zeros, then the system is always consistent, regardless of what the right hand side is.

If every column in the row echelon form of the coefficient matrix contains the leading entry of some row (in other words, if each leading entry is indented exactly one space as you move down the rows), then the linear system will never have more than one solution.
On the other hand, if some column does not contain the leading entry for any row, then this variable can be set arbitrarily and consequently if there is any solution at all, there will be infinitely many.

Another way of stating the second principle is that whether a linear system can have more than one solution or not depends on whether the **row echelon form** of the coefficient matrix has more columns than non-zero rows. (Note that the row echelon form could not possibly have fewer columns than non-zero rows. (WHY?))

In particular, if the original coefficient matrix has more columns than rows, then the system could never have only one solution. Applied to the homogeneous case, this can be stated as

Theorem 1.12 (p. 59). A homogeneous system of m linear equations in n unknowns always has a non-trivial solution if $m < n$.

Since the existence of multiple solutions (provided that there is any solution at all) depends only on the coefficient matrix and since a **homogeneous** system always has at least one solution (namely the trivial one), multiple solutions for a linear system are possible only if the corresponding homogeneous system has multiple solutions. But the homogeneous system has multiple solutions if and only if it has a **non-trivial solution.** In other words

A linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

which is **consistent** will have **more than one solution** or not depending on whether or not the corresponding homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

has a non-trivial solution.

If

$$A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad A_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the homogeneous system above can be written in the form

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = \mathbf{0}.$$

We recall that this does **not** have a non-trivial solution if and only if the vectors $A_1, \ldots, A_n$ are linearly independent. Therefore the assertion above can be rephrased as follows.

Theorem. A **consistent** system of linear equations will have a **unique** solution if and only if the columns of the coefficient matrix are linearly independent vectors.

Key words: Homogeneous. Solution. Trivial. Linearly independent. Consistent. Inconsistent. Linear combination.

Remember that **definitions** play the same role in Math 311 that formulas play in calculus. It is not enough just to know what a word means. In order to be able to write proofs, you need to know the **formal definitions** of the words.