M632 Spring 2009 Practice for the Prelim

Name:

Instructions: Do as many problems as you can. Complete solutions (except for minor flaws) to 4-5 problems would be considered a good performance.

1. Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 0$. Hint: think about continuous functions with compact support.

2. (a) State the Ascoli-Arzela theorem.
   (b) Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For $f \in C[0,1]$ define $Tf(x) = \int_0^1 K(x,y)f(y)\, dy$. Then $Tf \in C[0,1]$. Show that the closure of $\{Tf : ||f||_\infty \le 1\}$ is compact in $C[0,1]$.

3. Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{T}'$ be the *cocompact topology* on X. A set $O \subset X$ is open in $\mathcal{T}'$ if and only if O^c is compact in $\mathcal{T}$. Prove that $(X, \mathcal{T}')$ is Hausdorff if and only if $(X, \mathcal{T})$ is compact.

4. A metric space (X, d) is called *proper* if for every $x \in X$ and every $r > 0$, the closed ball $\{y \in X \mid d(y,x) \le r\}$ is compact. Suppose that (X, d) is a proper locally compact metric space. Prove that if a subset $K \subset X$ is compact then it is closed and bounded. In this context, bounded means that K is contained in a ball of finite radius.

5. (a) State the Weierstrass approximation theorem for continuous functions on $[0,1]$.
   (b) If μ is a regular Borel measure on $[0,1]$ and $\int_0^1 x^n\, d\mu = \frac{1}{n+1}$ for $n = 0, 1, 2, \ldots$, show that μ is Lebesgue measure.

6. Let m denote Lebesgue measure on $\mathbb{R}$ and define a measure μ by
$$\mu(E) = \int_E \frac{1}{1+x^2} dm.$$
   Show that m is absolutely continuous w.r.t. μ and compute $\frac{dm}{d\mu}$.

7. (a) State Fubini's theorem.
   (b) Prove that if f is a nonnegative measurable function on $\mathbb{R}$ and m is Lebesgue measure then for any Borel $E \subset \mathbb{R}$,
$$\int_E f(x)\, dx = \int_0^\infty m\big(\{x \in E \ : \ f(x) > y\}\big) dy.$$

8. Is the unit sphere in l^∞ compact in the weak* topology? The unit sphere is the set of all $x \in l^\infty$ such that $||x||_\infty = 1$. The weak* topology is obtained by identifying l^∞ as the dual space to l^1.

9. Let μ^* be an outer measure on a set X. Recall that this means $\mu^*(\emptyset) = 0$, μ^* is finitely additive and countably subadditive. Recall also that a set $E \subset X$ is μ^*-measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for every $A \subset X$. Prove that the set of all μ^*-measurable sets is a σ-algebra.

10. Let μ be a finite Borel measure on $\mathbb{R}$. Prove that μ is regular. Hint: use the Riesz-Markov theorem to obtain a regular measure ν such that $\int f d\mu = \int f d\nu$ for every continuous function f. Show that for every open interval (a, b), $\nu(a, b) = \mu(a, b)$ by approximating the characteristic function $\chi_{(a,b)}$ by continuous functions.