

Name:

Instructions: Do as many problems as you can. Complete solutions (except for minor flaws) to 4-5 problems would be considered a good performance.

1. Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{h \rightarrow 0} \int_{\mathbb{R}} |f(x+h) - f(x)| dx = 0$. Hint: think about continuous functions with compact support.

Solution. The continuous functions with compact support are dense in L^1 . Hence if $\epsilon > 0$ then there exists $g \in C_c(\mathbb{R})$ with $\|g - f\| < \epsilon$. Now,

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq \int_{\mathbb{R}} |f(x+h) - g(x+h)| + |g(x+h) - g(x)| + |g(x) - f(x)| dx.$$

g is uniformly continuous. So there exists $\delta > 0$ such that $|x - y| \leq \delta$ implies $|g(x) - g(y)| \leq \epsilon \|g\|_1^{-1}$. So if $|h| < \delta$ then $\int_{\mathbb{R}} |g(x+h) - g(x)| dx \leq \epsilon$. The above inequality now implies $\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq 3\epsilon$. Since ϵ is arbitrary this implies the result.

2. (a) State the Ascoli-Arzelà theorem.

Let \mathcal{F} be an equicontinuous family of functions from a separable metric space X to a metric space Y . Suppose that for each $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is compact. Then the closure of \mathcal{F} in the topology of uniform convergence on compact sets is sequentially compact. In fact, it is also compact since this topology is metrizable.

- (b) Let $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be continuous. For $f \in C[0, 1]$ define $Tf(x) = \int_0^1 K(x, y)f(y) dy$. Then $Tf \in C[0, 1]$. Show that the closure of $\{Tf : \|f\|_{\infty} \leq 1\}$ is compact in $C[0, 1]$.

Solution. By the Ascoli-Arzelà theorem it suffices to show that $\{Tf : \|f\|_{\infty} \leq 1\}$ is equicontinuous and for each $x \in [0, 1]$, $\{Tf(x) : \|f\|_{\infty} \leq 1\}$ is compact. Equicontinuity follows from $\|Tf_1 - Tf_2\|_{\infty} \leq \|K\|_{\infty} \|f_1 - f_2\|_{\infty} \leq 2\|K\|_{\infty}$ for all f_1, f_2 in the unit ball of L^{∞} .

The unit ball in L^{∞} is weak* compact. The functional $f \mapsto Tf(x)$ is weak* continuous because $K(x, y)$ is in L^1 (where x is fixed and y is the variable). So $\{Tf(x) : \|f\|_{\infty} \leq 1\}$ is the continuous image of a compact set. It is therefore compact.

3. Let (X, \mathcal{T}) be a Hausdorff space. Let \mathcal{T}' be the *cocompact topology* on X . A set $O \subset X$ is open in \mathcal{T}' if and only if O^c is compact in \mathcal{T} . Prove that (X, \mathcal{T}') is Hausdorff if and only if (X, \mathcal{T}) is compact.

Solution. Let $x, y \in X$ with $x \neq y$. If (X, \mathcal{T}') is Hausdorff then there exists disjoint \mathcal{T}' -open sets U, V such that $x \in U$ and $y \in V$. Then U^c and V^c are compact in \mathcal{T} .

Observe that $X = U^c \cup V^c = (U \cap V)^c$. Thus X is compact (because the union of any two compact sets is compact). If (X, \mathcal{T}) is compact and $O \subset X$ is \mathcal{T} -open then O^c is compact; hence O is \mathcal{T}' -open. Thus $\mathcal{T} = \mathcal{T}'$ and thus \mathcal{T}' is Hausdorff.

4. A metric space (X, d) is called *proper* if for every $x \in X$ and every $r > 0$, the closed ball $\{y \in X \mid d(y, x) \leq r\}$ is compact. Suppose that (X, d) is a proper locally compact metric space. Prove that if a subset $K \subset X$ is compact then it is closed and bounded. In this context, bounded means that K is contained in a ball of finite radius.

Solution. Every compact subset of a Hausdorff space is closed, so K is closed. Suppose for a contradiction that K is not contained in any ball of finite radius. Let x_0 be in K . Then there exists a point $x_1 \in K$ that is not in the radius 1 ball centered at x_0 . In general, if $x_0, x_1, \dots, x_n \in K$ then there exists a point $x_{n+1} \in K$ such that $d(x_{n+1}, x_i) \geq 1$ for all i with $0 \leq i \leq n$. This is because if no such point exists then it follows that K is contained in the ball of radius $1 + \sum_{i=1}^n d(x_0, x_i)$ centered at x_0 . Thus we have constructed an infinite sequence x_0, x_1, \dots such that $d(x_i, x_j) \geq 1$ for all $i \neq j$. But then $\{x_i\}$ is a sequence in K that has no convergent subsequence. This contradicts the compactness of K .

5. (a) State the Weierstrass approximation theorem for continuous functions on $[0, 1]$.

Solution. If $\mathcal{F} \subset C[0, 1]$ is any algebra that (1) contains the constant functions and (2) separates points then \mathcal{F} is dense in $C[0, 1]$ (in the norm topology).

- (b) If μ is a regular Borel measure on $[0, 1]$ and $\int_0^1 x^n d\mu = \frac{1}{n+1}$ for $n = 0, 1, 2, \dots$, show that μ is Lebesgue measure.

Solution. Let m denote Lebesgue measure. It follows that if p is any polynomial then $\int p d\mu = \int p dm$. By the Weierstrass approximation theorem, the set of polynomials is dense in $C[0, 1]$. Since μ is a Borel measure, the map $f \mapsto \int f d\mu$ on $C[0, 1]$ is continuous. Similarly, $f \mapsto \int f dm$ is continuous. Two continuous maps that agree on a dense set must be equal. Hence we have $\int f d\mu = \int f dm$ for all $f \in C[0, 1]$. By the uniqueness statement in the Riesz-Markov theorem this implies $\mu = m$. The regularity of μ does not have to be stated because every finite Borel measure on \mathbb{R} is automatically regular (see last problem below).

6. Let m denote Lebesgue measure on \mathbb{R} and define a measure μ by

$$\mu(E) = \int_E \frac{1}{1+x^2} dm.$$

Show that m is absolutely continuous w.r.t. μ and compute $\frac{dm}{d\mu}$.

Solution. This is a special case of one of the homework exercises. $\frac{dm}{d\mu}(x) = 1 + x^2$.

7. (a) State Fubini's theorem.

- (b) Prove that if f is a nonnegative measurable function on \mathbb{R} and m is Lebesgue measure then for any Borel $E \subset \mathbb{R}$,

$$\int_E f(x) dx = \int_0^\infty m(\{x \in E : f(x) > y\}) dy.$$

Solution. Let $G = \{(x, y) : f(x) > y \geq 0, x \in E\}$. Then

$$\begin{aligned} \int_E f(x) dx &= \int \chi_E(x) f(x) dx = \int \chi_G(x, y) dy dx \\ &= \int \chi_G(x, y) dx dy = \int_0^\infty m(\{x \in E : f(x) > y\}) dy. \end{aligned}$$

8. Is the unit sphere in l^∞ compact in the weak* topology? The unit sphere is the set of all $x \in l^\infty$ such that $\|x\|_\infty = 1$. Be careful, this is not the same as the unit ball. The weak* topology is obtained by identifying l^∞ as the dual space to l^1 .

Solution. The unit sphere is not weak* compact. For example, let $\{e_i\}$ be the standard basis vectors in l^∞ . These elements are all in the unit sphere. But $\lim e_i = 0$ in the weak* topology and 0 is not in the unit sphere.

9. Let μ^* be an outer measure on a set X . Recall that this means $\mu^*(\emptyset) = 0$, μ^* is finitely additive and countably subadditive. Recall also that a set $E \subset X$ is μ^* -measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for every $A \subset X$. Prove that the set of all μ^* -measurable sets is a σ -algebra.

This is in section 12.1 of Royden.

10. Let μ be a finite Borel measure on \mathbb{R} . Prove that μ is regular. Hint: use the Riesz-Markov theorem to obtain a regular measure ν such that $\int f d\mu = \int f d\nu$ for every continuous function f . Show that for every open interval (a, b) , $\nu(a, b) = \mu(a, b)$ by approximating the characteristic function $\chi_{(a,b)}$ by continuous functions.

Solution. Let $\Lambda : C(X) \rightarrow \mathbb{R}$ be the linear functional $\Lambda(f) = \int f d\mu$. Let ν be the Radon measure associated to Λ by the Riesz-Markov theorem. Let $O \subset \mathbb{R}$ be any open set. Then O is σ -compact; there exists compact sets K_i such that $O = \cup_{i=1}^\infty K_i$. Let f_n be a continuous function with $0 \leq f_n \leq 1$, $f_n = 1$ on $\cup_{i=1}^n K_i$ and on $\cup_{i=1}^{n-1} \text{supp}(f_i)$ and $f_n = 0$ on O^c . Then f_n increases monotonically to χ_O . The monotone convergence theorem implies

$$\mu(O) = \lim_n \int f_n d\mu = \lim_n \int f_n d\nu = \nu(O).$$

So μ and ν agree on open sets. If E is any Borel set and $\epsilon > 0$ then there exists an open $O \supset E$ a closed $K \subset E$ such that $\nu(O - K) < \epsilon$ (by regularity of ν). But $O - K$ is open so $\mu(O - K) < \epsilon$ too. This implies $\mu(O) < \mu(E) + \epsilon$ which implies outer regularity of μ . Hence by the uniqueness statement in the Riesz-Markov theorem, $\mu = \nu$.