We have developed the tools to solve the general quadratic equation

$$ax^2 + bx + c = 0; a, b, c \in \mathbb{R}$$

and will now discuss the solutions to equations that can be reduced to a quadratic equation.

Example Solve: $x^4 - 6x^2 + 8 = 0$

Answer Observe that $x^4 - 6x^2 + 8 = (x^2 - 4)(x^2 - 2) = 0$ and so either

$$x^2 - 4 = 0 \Rightarrow x = \pm 2 \qquad \text{or} \qquad x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$

and the solution set is $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$.

Notice that we could have used the quadratic formula if we changed the variable by setting $t = x^2$. Then our equation can be rewritten as

$$x^4 - 6x^2 + 8 = (x^2)^2 - 6(x^2) + 8 = t^2 - 6t + 8 = 0$$

and we apply the quadratic formula with $a = 1, b = -6, c = 8$ and obtain

$$t = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm 2}{2} \Rightarrow t \in \{2, 4\}$$

We have thus solved the reduced equation, but $\{2, 4\}$ is not the solution set to our original problem. Remember that $t = x^2$ so as above we need to solve $x^2 = 2$ and $x^2 = 4$, which will produce the set of solutions to the original equations, i.e. $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$.

Example Rewrite each equation of quadratic type as a quadratic equation in t and give the change of variable.

1. $x^6 + 2x^3 + 1 = 0$
2. $(x - 2)^4 + (x - 2)^2 + 6 = 0$
3. $8x^{\frac{4}{3}} + 2x^{\frac{2}{3}} + 1 = 0$
4. $x^{-6} + x^{-3} + 7 = 0$
5. $2^{4x} + 4^x + 3 = 0$

Answer

1. $t^2 + 3t + 1 = 0, t = x^3$
2. $t^2 + t + 6 = 0, t = (x - 2)^2$
3. $8t^2 + 2t + 1 = 0, t = x^{\frac{2}{3}}$
4. $t^2 + t + 7 = 0, t = x^{-3}$
5. $t^2 + t + 3 = 0, t = 2^{2x}$

Example Find all real solutions of each equation.

1. $y^{-2} - \frac{7}{y} + 12 = 0$
2. $x^{\frac{2}{3}} = 9$
3. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 28 = 0$

Answer

1. Rewrite $y^{-2}-\frac{7}{y}+12 = 0$ with the change of variable $t = y^{-2}$ and obtain $t^2-7t+12 = 0$. This reduced equation factors as $(t-3)(t-4) = 0$ and whence its solution set it $\{3, 4\}$. The solutions to the original equation is obtained by solving $y^{-2} = t = 3$ and $y^{-2} = t = 4$.
$$y^{-2} = 3 \Rightarrow y^2 = \frac{1}{3} \Rightarrow y = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$
$$y^{-2} = 4 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$$
Therefore, the solution set is $\{-\frac{1}{2}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{1}{2}\}$

2. Rewrite $x^{\frac{2}{3}} = 9$ with the change of variable $t = x^{\frac{1}{3}}$ and obtain $t^2 = 9$. It is clear that the set of solution to the reduced equation is $\{-3, 3\}$. Then to obtain the solutions to the original equation we solve
$$x^{\frac{1}{3}} = t = -3 \Rightarrow x = -27$$
$$x^{\frac{1}{3}} = t = 3 \Rightarrow x = 27$$
Therefore, the solution set is $\{-27, 27\}$.

3. Rewrite $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 28 = 0$ with the change of variable $t = x^{\frac{1}{3}}$ and obtain $t^2 + 3t - 28 = 0$. Observe that $t^2 + 3t - 28 = (t+7)(t-4) = 0$ and therefore the solution set to the reduced equation is $\{-7, 4\}$. Then
$$x^{\frac{1}{3}} = t = -7 \Rightarrow x = -343$$
$$x^{\frac{1}{3}} = t = 4 \Rightarrow x = 64$$
and the original equation has the solutions $\{-343, 64\}$.