4. (a) \(153 + 102 + 55 + 202 + 108 + 20 + 110 + 160 + 175 + 155 = 1240\) votes
(b) \(\frac{1240}{2} = 620\), so 621 needed for majority
(c) Brandy has more first-place votes.

6. initial election: A has 310 first place votes
   B has 330
   C has 270
   D has 330
(a) 25% of 1240 is 310, so C is eliminated.
(b) new schedule:

<table>
<thead>
<tr>
<th></th>
<th>255</th>
<th>310</th>
<th>180</th>
<th>175</th>
<th>155</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>D</td>
<td>D</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>D</td>
<td>B</td>
<td>A</td>
<td>D</td>
<td>B</td>
</tr>
</tbody>
</table>

(c) still no winner after recount. Also, each of A, B, D has more than 25% of vote, so this voting method is now deadlocked!

12. (a) first-place votes: A - 16, B - 16, C - 11, D - 12
    A, B tie for win
(b) last-place votes: A - 17, B - 15
    B wins the tie
(c) head-to-head: A vs. B
    \(\frac{24}{31}\) A wins the tie
14. (a) after $120/150$ votes, A - 26
    B - 18
    C - 42
    D - 34

    B needs 24 to tie with C. This would mean A - 26
    B - 42
    C - 42
    D - 34

    with 6 votes remaining, B needs 4/6 of these
    votes to ensure that B beats C.

    So B needs $24 + 4 = 28$ of the 30 remaining votes.

(b) Likewise, D needs 8 to bring it to a tie.

    The A - 26
    B - 18
    C - 42
    D - 42

    with 22 votes remaining, D needs
    12 of these to guarantee a win.

    So D needs $8 + 12 = 20$ of the 30
    remaining votes.

16. (a) $1025/4 = 256.25$, so 257 votes minimum for plurality

(b) $$1975$$

    guess-and-check, notice
    $$\frac{1025}{8} = 128, 125 \approx 129\text{ votes needed}$$

(c) same method, $\frac{1025}{5} = 205 \approx 206\text{ needed}$

    so 5 candidates
24. (a) Max is 20 first-place votes: \(20 \times 5 = 100\) points
(b) Min in 20 last-place votes: 20 points
(c) \(5 + 4 + 3 + 2 + 1 = 15\) points
(d) \(20 \times 15 = 300\) points
(e) \(300 - 69 - 70 - 64 - 48 = 49\) points

26. 40 voters, 15 pts each \(\rightarrow 600\) points total.
\(600 - 139 - 121 - 80 - 113 = 147\) points for E.

28. First round: 1240 total votes, 621 needed for majority.
First-place votes: A has 310
B has 330
C has 270
D has 330
No majority, C gets eliminated. New table is:

<table>
<thead>
<tr>
<th>(255)</th>
<th>165</th>
<th>202</th>
<th>128</th>
<th>160</th>
<th>175</th>
<th>155</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>C</td>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>counted these up in #18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First-place votes:
A has 420
B has 490
D\&E has 330
No majority, D gets eliminated. New table is:

<table>
<thead>
<tr>
<th>(255)</th>
<th>165</th>
<th>202</th>
<th>128</th>
<th>160</th>
<th>175</th>
<th>155</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

Out combining, this is:

<table>
<thead>
<tr>
<th>(255)</th>
<th>165</th>
<th>202</th>
<th>128</th>
<th>160</th>
<th>175</th>
<th>155</th>
</tr>
</thead>
<tbody>
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<td>B</td>
<td>A</td>
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<td>2</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

\(B\) wins
34. 17 votes, need 9 for majority.

(a) First-place votes: 
\[ \begin{array}{cccc}
A & B & C & D \\
5 & 5 & 6 & 0 \\
\end{array} \]
no majority, D gets the last

Next round:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

First-place votes: 
\[ \begin{array}{cccc}
A & B & C & D \\
5 & 5 & 6 & 0 \\
\end{array} \]
no majority

B is eliminated

C wins.

Next round:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

(c) First-place votes: 
\[ \begin{array}{cccc}
A & B & C & D \\
5 & 5 & 6 & 2 \\
\end{array} \]
no majority, C eliminated

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>2</td>
<td>D</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>B</td>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>4</td>
<td>C</td>
<td>D</td>
<td>B</td>
</tr>
</tbody>
</table>

next round:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>2</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

no majority, A eliminated

next round:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

B wins

(c) this violates Monotonicity Criterion
(a) | A | B | C | D |
---|---|---|---|---|
1  | 10 | 11 | 4 | 2 |
2  | 11 | 10 | 2 | 4 |
3  | 2  | 4  | 11 | 10 |

C is Condorcet candidate

(b) No initial majority, C is eliminated.

Next round:

|   | 10 | 11 | 4 | 2 |
---|----|----|---|---|
1  | A  | B  | D | D |
2  | D  | D  | B | A |
3  | B  | A  | A | B |

No majority, D is eliminated

Next round:

|   | 12 | 15 |
---|----|----|
1  | A  | B  |
2  | B  | A  |

B is winner.

(c) If D drops out,

|   | 10 | 11 | 4 | 2 |
---|----|----|---|---|
1  | A  | B  | C | A |
2  | C  | C  | B | B |
3  | B  | A  | A | C |

No majority, B is eliminated

Next round:

|   | 12 | 15 |
---|----|----|
1  | A  | C  |
2  | C  | A  |

C is winner.

(d) Since in (b) the Condorcet candidate lost, it violates the Condorcet Condition.

Since D dropping out changed the outcome, it violates the IIA Criterion.

(Also it's not shown here, but it also violates Monotonicity)
62. This was my thought process:

Since $C$ must be a Condorcet candidate, but receives no first-place votes, put $C$ second on every ballot.

\[
\begin{array}{c|cccc}
1^\text{st} & A & B & D \\
2^\text{nd} & C & C & C \\
3^\text{rd} & \\
4^\text{th} & \\
\end{array}
\]

Since nobody has a majority, come up with amounts that give no majority. Having 100 total votes seems easy. Also, $A$ is the plurality candidate and $D$ wins no elections, so give these the most/least of the first-place votes.

\[
\begin{array}{c|c|c|c}
1^\text{st} & 50 & 49 & 1 \\
2^\text{nd} & A & B & D \\
3^\text{rd} & C & C & C \\
4^\text{th} & \\
\end{array}
\]

We want $B$ to win the Borda count, so stack it high in the remaining places, while giving $A$ a bad position (since the plurality candidate may be a contender in the Borda count).

\[
\begin{array}{c|c|c|c}
1^\text{st} & 50 & 49 & 1 \\
2^\text{nd} & A & B & D \\
3^\text{rd} & C & C & C \\
4^\text{th} & B & D & B \\
\end{array}
\]

How does this work? Check the conditions:

(i) There is no majority candidate.

(ii) Checking the Condorcet pairings, we notice that $A \text{ v } C$ is a tie. So change the amounts to give $C$ a better standing.

\[
\begin{array}{c|c|c|c}
1^\text{st} & 49 & 48 & 3 \\
2^\text{nd} & A & B & D \\
3^\text{rd} & C & C & C \\
4^\text{th} & B & D & B \\
\end{array}
\]
Now check the conditions with this new election:

(i) There is no majority candidate.

(ii) We think that $C$ is the Condorcet candidate, so check those, and we see that $C$ wins all three of its pairings. So $C$ is indeed the Condorcet candidate.

(iii) Using Borda count, we find that $A$ has 247 points, $B$ has 296 points, $C$ has 300 points, and $D$ has 157 points. Close, but no cigar!

We wanted $B$ to win under this method, so we need to hurt $C$ a little bit. We can do this without changing the other conditions by splitting the second column into two:

<table>
<thead>
<tr>
<th></th>
<th>49</th>
<th>40</th>
<th>8</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$A$</td>
<td>$B$</td>
<td>$B$</td>
<td>$D$</td>
</tr>
<tr>
<td>2nd</td>
<td>$C$</td>
<td>$C$</td>
<td>$D$</td>
<td>$C$</td>
</tr>
<tr>
<td>3rd</td>
<td>$B$</td>
<td>$D$</td>
<td>$C$</td>
<td>$B$</td>
</tr>
<tr>
<td>4th</td>
<td>$D$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
</tr>
</tbody>
</table>

This gives $D$ a few more Borda points, $C$ a few less, and doesn’t change the Borda score of $A$ and $B$ since their standings didn’t change. It changes the Condorcet pairings between $C$ and $D$, but $C$ still comes out on top. Check the conditions again:

(i) There is no majority candidate.

(ii) Candidate $C$ is still the Condorcet candidate.

(iii) Now, under Borda count $A$ has 247 points, $B$ has 296 points, $C$ has 292 points, and $D$ has 165 points. ¡Perfecto!

(iv) Finally, $A$ is the plurality candidate.

So this configuration works. Another possible configuration (I looked this up later in the textbook—it was very similar to the above, but has easier numbers):

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$A$</td>
<td>$B$</td>
<td>$B$</td>
<td>$D$</td>
</tr>
<tr>
<td>2nd</td>
<td>$C$</td>
<td>$D$</td>
<td>$C$</td>
<td>$C$</td>
</tr>
<tr>
<td>3rd</td>
<td>$B$</td>
<td>$C$</td>
<td>$D$</td>
<td>$B$</td>
</tr>
<tr>
<td>4th</td>
<td>$D$</td>
<td>$A$</td>
<td>$A$</td>
<td>$A$</td>
</tr>
</tbody>
</table>
64. If candidate A is winning under the Borda count method, this candidate has more points than any other. A change in the election benefiting only A can only increase A’s points and decrease the points of the other candidates. So the winner of the election will remain A.

Note: I accidentally wrote a solution to #65 when I meant to write a solution to #64. So here is a solution to #65, although it was not assigned.

65. To explain this, first suppose that X was winning against Y in the original election. In the reelection, the changes favor only X. So if there are any changes in the comparison of X and Y, they benefit X only. Therefore X is still winning against Y in the reelection.

Next, suppose that X was losing against Z in the original election. Again, the changes can only benefit X, so now X is maybe still losing, or is perhaps now tied or winning against Z in the reelection.

So, the number of pairwise comparisons that X wins in the reelection can only improve—this number cannot go down. Since the changes do not change the relative orders of the other candidates, the numbers of comparisons that each of them win can only decrease. They may now be losing the comparison against X, and all other comparisons are the same as they were in the original election.

68. First of all, adding in the average makes no difference. By averaging the points, each ranking is divided by the total number of candidates. If rank x is higher or lower than rank y, it will remain so after dividing by some positive number. So now we need to see why adding up ranks is the same as counting Borda points.

Let’s assume we have an election with 10 candidates. We want to compute the Borda count for candidate A. Well, we look at the first ballot, and see that A was in third place, so A gets 8 points. Another way to write this is A gets 8 = 1 + 10 = 3 points. On the next ballot, A got first, so 10 = 1 + 10 − 1 points are scored. In general, if A is in nth place on a ballot, 1 + 10 − n is added to A’s total points. If A is in n1th place on the first ballot, n2th place on the second ballot, and so on, we can write A’s score on the first five ballot as: (1 + 10 − n1) + (1 + 10 − n2) + (1 + 10 − n3) + (1 + 10 − n4) + (1 + 10 − n5). Using a little algebra, this is: 5*11 − (n1 + n2 + n3 + n4 + n5). So we are just subtracting A’s rank on each ballot from a total amount. In other words, the lowest sum of ranks on ballots is giving the highest Borda count score. Therefore the two voting methods are equivalent.