

Convergence of Series

Marvin Ortel

I. General Theorems

The following two theorems, which we do not prove, are fundamental for the study of convergence of series. Throughout this article we use the following standard notations: $\mathbb{N} \equiv \{0, 1, 2, 3, \dots\}$; \mathbb{R} is the set of real numbers.

Theorem 1. Suppose that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and $M \in \mathbb{R}$, and that $\sum_{n=0}^N |a_n| \leq M$ for all $N \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} |a_n|$ converges and

$$0 \leq \sum_{n=0}^{\infty} |a_n| \leq M.$$

Theorem 2. Suppose that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and that $\sum_{n=0}^{\infty} |a_n|$ converges. Then $\sum_{n=0}^{\infty} a_n$ converges and

$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=0}^{\infty} |a_n|.$$

The next two theorems provide, perhaps, the most useful test for the convergence of series.

Theorem (Basic ratio test for convergence). Suppose that $r \in [0, 1)$ and that for all $n \in \mathbb{N}$ we have $a_n \in \mathbb{R} \setminus \{0\}$ and $\frac{|a_{n+1}|}{|a_n|} \leq r$. Then

$$\sum_{n=0}^N |a_n| \leq \frac{|a_0|}{1-r}$$

for all $N \in \mathbb{N}$, and $\sum_{n=0}^{\infty} |a_n|$ converges.

Proof. Choose $N \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{n=0}^N |a_n| &= \sum_{n=0}^N |a_0| \frac{|a_1|}{|a_0|} \frac{|a_2|}{|a_1|} \cdots \frac{|a_n|}{|a_{n-1}|} \leq \\ &\leq \sum_{n=0}^N |a_0| r^n = |a_0| \frac{1-r^{N+1}}{1-r} \leq \frac{|a_0|}{1-r}. \end{aligned}$$

Thus $\sum_{n=0}^N |a_n| \leq \frac{|a_0|}{1-r}$ for all $N \in \mathbb{N}$. Thus, by Theorem 1, with $M \equiv \frac{|a_0|}{1-r}$, it follows that $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem (General ratio test for convergence). Suppose that $r \in [0, 1)$, that $Q \in \mathbb{N}$, that $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and that $a_Q \neq 0$. Also suppose that $\frac{|a_q|}{|a_p|} \leq r$ whenever the following five conditions hold: (i) $p, q \in \mathbb{N}$; (ii) $Q \leq p < q$; (iii) $a_p \neq 0$; (iv) $a_q \neq 0$; (v) $a_k = 0$ whenever $k \in \mathbb{N}$ and $p < k < q$. Then for all $n \in \mathbb{N}$ we have that $\sum_{n=0}^N |a_n| \leq$

$$\sum_{n=0}^{Q-1} |a_n| + \frac{|a_Q|}{1-r}, \text{ and the series } \sum_{n=0}^{\infty} |a_n| \text{ converges.}$$

Proof. Choose $N \in \mathbb{N}$. Define $c_0 \equiv a_Q$, and, for each $k \in \mathbb{N}$ with $k > 1$ let c_k be the k -th non-zero term of the sequence (a_n) that appears after a_Q . Then

$$\sum_{n=0}^N |a_n| \leq \sum_{n=0}^{Q-1} |a_n| + \sum_{k=0}^N |c_k|.$$

By the basic ratio test, we have

$$\sum_{k=0}^N |c_k| \leq \frac{|c_0|}{1-r} = \frac{|a_Q|}{1-r}.$$

The following two theorems give tests for the divergence of series.

Theorem. Suppose that $\epsilon > 0$, and $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and S is an infinite subset of \mathbb{N} , and that $|a_s| \geq \epsilon$ whenever $s \in S$. Then the series $\sum_{n=0}^{\infty} a_n$ diverges.

Proof. We argue by contradiction. Thus, suppose that $\sum_{n=0}^{\infty} a_n = L$

and choose $N \in \mathbb{N}$ so that $\left| \sum_{k=0}^n a_k - L \right| \leq \epsilon/4$ whenever $n \in \mathbb{N}$ and $n \geq N$. Choose $s \in S$ such that $s > N$. Then

$$\begin{aligned} \epsilon \leq |a_s| &= \left| \sum_{k=0}^s a_k - \sum_{k=0}^{s-1} a_k \right| = \\ &= \left| \left(\sum_{k=0}^s a_k - L \right) - \left(\sum_{k=0}^{s-1} a_k - L \right) \right| \leq \\ &\leq \left| \sum_{k=0}^s a_k - L \right| + \left| \sum_{k=0}^{s-1} a_k - L \right| \leq \epsilon/4 + \epsilon/4 \end{aligned}$$

which is a contradiction.

Theorem. Suppose that $r > 1$, and $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and (s_k) is an increasing sequence from \mathbb{N} , and that

$$\frac{|a_{s_{k+1}}|}{|a_{s_k}|} \geq r$$

whenever $k \in \mathbb{N}$. Then the series $\sum_{n=0}^{\infty} a_n$ diverges.

Proof. Choose $\epsilon > 0$ and choose $K \in \mathbb{N}$ such that $|a_{s_0}|r^k > \epsilon$ whenever $k \in \mathbb{N}$ and $k \geq K$. Then if $k \in \mathbb{N}$ and $k \geq K$, we have that

$$|a_{s_k}| = |a_{s_0}| \frac{|a_{s_1}|}{|a_{s_0}|} \frac{|a_{s_2}|}{|a_{s_1}|} \cdots \frac{|a_{s_k}|}{|a_{s_{k-1}}|} \geq |a_{s_0}|r^k > \epsilon.$$

Thus $|a_t| > \epsilon$ for all t in the infinite set $\{s_K, s_{K+1}, s_{K+2}, \dots\}$. Thus, the series $\sum_{n=0}^{\infty} a_n$ diverges.