General solutions of Differential Equations

The following is a possible question on Test IV:

"State a theorem which describes the general solutions
of certain linear differential equations with constant cofficients."

The following text is considered to be a correct answer to that question.

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Theorem 1. Suppose that $b, c, r \in \mathbb{R}$, $b \neq 0$, r is the root of $bX + c$, I is an open interval, $f : I \to \mathbb{R}$, and ψ is a particular solution of the differential equation

$$\text{(DE)} \quad by'(t) + cy(t) = f(t).$$

Then the general solution of DE is

$$y(t) = \psi(t) + Ae^{rt}.$$

Theorem 2. Suppose that $a, b, c, r, s, \alpha, \beta \in \mathbb{R}$, $a \neq 0$, $\beta \neq 0$, I is an open interval, $f : I \to \mathbb{R}$, and ψ is a particular solution of the differential equation

$$\text{(DE)} \quad ay''(t) + by'(t) + cy(t) = f(t).$$

Then the following statements are true:

(i) If $r \neq s$ and r, s are roots of $aX^2 + bX + c$, then the general solution of DE is

$$y(t) = \psi(t) + Ae^{rt} + Be^{st}.$$

(ii) If r is a root of $aX^2 + bX + c$ of multiplicity 2, then the general solution of DE is

$$y(t) = \psi(t) + Ae^{rt} + Bte^{rt}.$$

(ii) If $\alpha + i\beta$ is a root of of $aX^2 + bX + c$, then the general solution of DE is

$$y(t) = \psi(t) + Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t).$$

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