Roots of a Polynomial

The following is a possible question on Test IV:

"Define the expression 'z is a root of $P(X)$ of multiplicity M' ".

The following text is considered to be a correct answer to that question.

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Definition. The expression "z is a root of $P(X)$ of multiplicity M" means that $P(X)$ is a polynomial, $z \in \mathbb{C}$, $M \in \mathbb{N}$, $(X - z)^M$ divides $P(X)$, and $(X - z)^{M+1}$ does not divide $P(X)$.

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Remark. In this terminology, the special case $M = 0$ is particularly important because of the following theorem.

Theorem. z is a root of $P(X)$ of multiplicity 0 if and only if z is NOT A ROOT of $P(X)$.

Proof. First suppose that z is a root of $P(X)$ of multiplicity 0. Then $(X - z)^0 = 1$ divides $P(X)$ and $(X - z)^{0+1} = X - z$ does not divide $P(X)$. Hence $X - z$ does not divide $P(X)$ and, by the division theorem, z is not a root of $P(X)$.

Now suppose that z is not a root of $P(X)$. Then, by the division theorem, $X - z$ does not divide $P(X)$. Also, the polynomial $(X - z)^0 = 1$ divides every polynomial. Hence $(X - z)^0$ divides $P(X)$. Thus z is a root of $P(X)$ of multiplicity 0.