Time-Minimal Control of Dissipative Two-level Quantum Systems: the Generic Case

Bernard Bonnard, Monique Chyba and Dominique Sugny

Abstract—The objective of this article is to complete preliminary results from [5], [18] concerning the time-minimal control of dissipative two-level quantum systems whose dynamics is governed by the Lindblad equation. The extremal system is described by a 3D-Hamiltonian depending upon three parameters. We combine geometric techniques with numerical simulations to deduce the optimal solutions.

Index Terms—Time optimal control, conjugate and cut loci, quantum control

I. INTRODUCTION

In this article, we consider the time-minimal control analysis of two-level dissipative quantum systems whose dynamics is governed by the Lindblad equation. More generally, according to [12], [14], the dynamics of a finite-dimensional quantum system in contact with a dissipative environment is described by the evolution of the density matrix \( \rho \) which is a positive semidefinite Hermitian operator having tr\((\rho)\) = 1 and tr\((\rho^2)\) \leq 1 [11], [12], [14]. The evolution of \( \rho \) is given by

\[
\frac{d\rho}{dt} = [H_0 + H_1, \rho] + iL(\rho),
\]

where \( H_0 \) is the field-free Hamiltonian of the system, \( H_1 \) represents the interaction with the control field and \( L \) the dissipative part of the equation; \([A,B]\) is the commutator of the operators \( A \) and \( B \) defined by \([A,B] = AB - BA\).

Equation (1) is written in units such that \( \hbar = 1 \). In the eigenbasis of \( H_0 \), the components of the density matrix satisfy the following equations:

\[
\dot{\rho}_{nn} = -i[H_0 + H_1, \rho]_{nn} - \sum_{k \neq n} \gamma_{kn} \rho_{kn} + \sum_{k \neq n} \gamma_{nk} \rho_{nk},
\]

where 1 \( \leq k \leq N \) and 1 \( \leq n \leq N \) for an \( N \)-level quantum system. The parameters \( \gamma_{kn} \) describe the population relaxation from state \( k \) to state \( n \) whereas \( \Gamma_{kn} \) is the dephasing rate of the transition from state \( k \) to state \( n \). Note that not every positive parameter \( \gamma_{kn} \) or \( \Gamma_{kn} \) is acceptable from a physical point of view due to the properties of the density matrix.

The analysis is motivated by physical reasons. We consider a fundamental model in quantum control extending the control of quantum systems in the conservative case [7] and describing a variety of physical systems. An example is given by the control of molecular alignment by laser fields in dissipative media where dissipation effects are due to molecular collisions. For a diatomic molecule driven by a linearly polarized laser field, molecular alignment means an increased probability distribution along the polarization axis. In this context, it has been proposed theoretically [15] and confirmed experimentally that such systems are governed by the Lindblad equation [20].

For molecular alignment, the dimension of the Hilbert space is infinite but can be truncated to a finite one if the intensity of the laser field is sufficiently weak. In standard experiments on molecular alignment, about twenty modes have generally to be taken into account.

In this article, we consider as a first step the control of two-level quantum systems controlled by laser fields. It allows to make a neat geometric analysis and it can be used as test case for our numerical computations based on the Cotcot-code [3] (including shooting methods and second order optimality conditions) and continuation methods on the set of parameters. Also this case describes the dynamics of a spin 1/2 particle in a magnetic field [10].

Hence, Partializing now equation (1) to the case \( N = 2 \), we assume that \( H_1 \) is of the form

\[
H_1 = -\mu_x E_x - \mu_y E_y,
\]

where the operators \( \mu_x \) and \( \mu_y \) are proportional to the Pauli matrices \( \sigma_x \) and \( \sigma_y \) in the eigenbasis of \( H_0 \). The electric field is the superposition of two linearly polarized fields \( E_x \) and \( E_y \) and we assume that these two fields are in resonance with the Bohr frequency \( E_2 - E_1 \). For a spin 1/2 particle, we can consider the same Hamiltonian where \( H_0 \) corresponds to a constant magnetic field along the \( z \)-axis and the dynamics is controlled by two magnetic fields \( B_x \) and \( B_y \) polarized respectively along the \( x \)- and \( y \)-axis.

In the Rotating Wave Approximation which consists in an averaging procedure over the rapid oscillations of the field [13], the time evolution of \( \rho(t) \) satisfies the following form of the Lindblad equation \( \frac{d}{dt}\rho = M\rho \) where \( \rho \) is written as a vector \((\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})\) and

\[
M = \begin{pmatrix}
-\gamma_{12} & -E^* & E & i\gamma_{21} \\
-E & -\omega - i\Gamma & 0 & E \\
E^* & 0 & \omega - i\Gamma & -E^* \\
i\gamma_{12} & E^* & -E & -i\gamma_{21}
\end{pmatrix}.
\]

The field \( E \) is equal to \( E = \omega e^{i\omega t}/2 \) where \( \omega \) is the complex Rabi frequency of the laser field (the real and imaginary parts of \( \omega \) are the amplitudes of the real fields \( E_x \) and \( E_y \) up to a multiplicative constant). The frequency \( \omega \) is the difference of energy between the ground and excited states of the system.
In the interaction representation, $M$ becomes

$$
\dot{M} = \begin{pmatrix}
-i\gamma_{12} & -u^*/2 & u/2 & i\gamma_{21} \\
-u/2 & -i\Gamma & 0 & u/2 \\
u^*/2 & 0 & -i\Gamma & -u^*/2 \\
i\gamma_{12} & u^*/2 & -u/2 & -i\gamma_{21}
\end{pmatrix}.
$$

(3)

The interaction representation means that we have transformed the mixed-state $\rho$ with the unitary transformation $U = \text{diag}(1, e^{i\omega_1 t}, e^{-i\omega_1 t}, 1)$. The new matrix $\dot{M}$ is then given by $\dot{M} = U^{-1}MU - iU^{-1}dU/dt$. Since $\text{Tr}[\rho] = 1$ and $\rho = \rho_1$, the $N \times N$ coefficients representing the density matrix can be replaced by $N^2 - 1$ real parameters. For a two-level quantum system, the density matrix $\rho$ can be represented by the vector $\rho = (x, y, z)$ where $x = 2\Re[\rho_{12}], y = 2\Im[\rho_{12}]$, and $z = \rho_{22} - \rho_{11}$ and $\rho$ belongs to the Bloch ball $|\rho| \leq 1$. The Lindblad equation takes the form:

$$
\begin{cases}
\dot{x} = -\Gamma x + u_2 z \\
\dot{y} = -\Gamma y - u_1 z \\
\dot{z} = (\gamma_{12} - \gamma_{21}) - (\gamma_{12} + \gamma_{21})z + u_1 y - u_2 x
\end{cases}.
$$

(4)

$\Lambda = (\Gamma, \gamma_+, \gamma_-)$ is the set of parameters such that $\gamma_+ = \gamma_{12} + \gamma_{21}$ and $\gamma_- = \gamma_{12} - \gamma_{21}$ and they satisfy the following inequalities $2\Gamma \geq \gamma_+ \geq \gamma_- \geq 0$ derived from the Lindblad equation [17], the Bloch ball $|\rho| \leq 1$ being invariant. The distance to the origin of the ball represents the purity of the system. A point of the Bloch sphere corresponds to a pure state. The control is $u = u_1 + iu_2$ where $|u| \leq M$ and up to a rescaling of the time and dissipative parameters we can assume that $|u| \leq 1$. It is a neat geometric representation, in which the state of the system is identified to a point of the unit ball, the drift term represents the dissipation and the components of the laser fields correspond to rotations along the $x$ and $y$ axes. Also this guides the representation of the system using spherical coordinates: $\rho$ denotes the distance to the origin, $\theta$ is the angle of revolution around the $z$-axis and $\phi$ is the angle along the meridian.

If we consider the optimal control problems, the costs important for applications are the time minimal transfer or the energy minimization and we shall concentrate on the first problem (both problems share similar geometric properties and can be handled using the same techniques but in the time-minimal case the control constraints can be taken into account directly and the geometric analysis is more striking). Hence, we have to analyze a time-minimal control problem for a bilinear system of the form:

$$
\dot{q} = F_0(q) + \sum_{i=1}^2 u_i F_i(q), \quad |u| \leq 1,
$$

where the drift term $F_0$ depends upon three parameters. This is a very difficult problem whose analysis requires advanced mathematical tools from geometric control theory and numerical simulations. Numerous geometric optimal control results exist in the conservative case e.g. [7], but only partial ones for this problem: a pioneering work [18] assuming $u$ real and a second one [5] for $u$ complex but restricted to $\gamma_- = 0$. Using spherical coordinates, outside the switching surface $\Sigma : p_\phi = p_\theta \cot \phi = 0$, the extremal curves are smooth solutions of the Hamiltonian differential equation defined by:

$$
H_\gamma = [\gamma_- \cos \phi - \rho(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)]p_\rho + p_\phi \left[\frac{\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2}(\gamma_+ - \Gamma)\right] + \sqrt{p_\rho^2 + p_\phi^2 \cot^2 \phi}.
$$

In the case $\gamma_- = 0$, the extremal system is integrable and we have obtained in [5] a complete geometric analysis based on a geometric continuation method on the set of dissipation parameters starting from the case $\gamma = \gamma_+$, where the analysis is reduced to a Riemannian problem on a two-sphere of revolution for the metric $g = d\phi^2 + \sin^2 \phi d\theta^2$. The general case where $\gamma_- \neq 0$ cannot be deduced from the integrable case, although some geometric properties persist: two types of asymptotic behaviors for the extremal trajectories connected to the existence or not of conjugate points, which will guide our computations. Indeed our analysis will show bifurcations phenomena for the accessibility set which distinguishes the integrable case from the generic case. The aim of this article is to complete the integrable case and to make a complete study for every generic parameter in $\Lambda$, combining mathematical reasoning and intensive numerical simulations using shooting techniques and including computations of conjugate points to test optimality. Based on the Cotcot code [3], they can be used in practice to compute the true optimal control, once the physical parameters are identified.

The organization of this article is the following. In section II, we complete the classification of the time-minimal synthesis of [18] corresponding to the case where $u$ is real but introducing more general tools to handle the problem. It corresponds to a time-minimal control problem of a two-dimensional bilinear system in the single-input case. The optimal synthesis for a fixed initial point can be constructed by gluing together local optimal syntheses. We can also make estimates of switching points by lifting the system on a semi-direct product Lie group. This classification is physically relevant to analyze the 3D-case because, using the symmetry of revolution of the problem, it gives the time-minimal synthesis for initial points $q_0 = (0, 0, \pm 1)$. This geometric property is explained in section III. Moreover, using spherical coordinates the system can be viewed as a system on a two-sphere of revolution coupled with the evolution of the distance to the origin, which represents the purity of the system. According to the maximum principle, smooth extremals are solutions of the Hamiltonian vector field $\vec{H}_r$ where $H_r = H_0 + (H_1^2 + H_2^2)^{1/2}$, $H_i = (p, F_i(q))$, and the control components are given by $u_i = H_i / (H_1^2 + H_2^2)^{1/2}$, $i = 1, 2$. Non smooth extremals can be constructed by connecting smooth subarcs of the switching surface $\Sigma; H_1 = H_2 = 0$. In general this problem is technically very difficult to analyze and the existence of the switching surface can lead to complicated behaviors for the extremal trajectories. A contribution of this article in section III is to classify the possible connections in our problem. We proved that every non smooth extremal is either a solution of the 2D-single input system, assuming $u$ real, or occurs when meeting the equatorial plane of the Bloch ball. In the second case, the switching can be handled numerically using an integrator with an adaptative step. In the same section,
we combine analytical and numerical analysis to determine the extremas and compute conjugate points. This completes the analysis from [5] in the integrable case. The physical interpretation is presented as a conclusion.

II. THE 2D-CASE

Following [18], a first step in the analysis is to consider the following reduced system. Assuming \( u \) real, the \( x \)-coordinate is not controllable and we can consider the planar single-input system:

\[
\begin{align*}
\dot{y} &= -\Gamma y - u_1 z \\
\dot{z} &= \gamma_+ z + u_1 y, \quad |u_1| \leq 1.
\end{align*}
\]  

(5)

The parameters satisfy the following inequalities

\[2\Gamma \geq \gamma_+ \geq |\gamma_-|.

The problem is to make the time-optimal analysis of the system when the initial state \( q(0) = (y(0), z(0)) \) is a pure state on the \( z \)-axis, that is \( q(0) = (0, \pm 1) \). Using a discrete symmetry group associated to reflections with respect to the two axes, one can assume that the initial point is \((0, 1)\) and moreover we can restrict our analysis to the domain \( y \geq 0 \). This is connected to the symmetry of revolution of the whole system around the \( z \)-axis.

We proceed as follows to make the analysis.

A. The feedback classification

Due to the complexity of the study, a first step in our analysis is to consider the feedback classification problem. The system is written in a more compact form as follows:

\[\dot{q} = F(q) + uG(q)\]

where \( F \) and \( G \) are affine vector fields. To make the feedback classification, we relax the control bound \( |u| \leq 1 \). According to [4], the geometric invariants are related to the sets:

- The singular set: \( S = \{q, \det(G, [F, G]) = 0\} \) where \( [\cdot, \cdot] \) are located the singular trajectories.
- The collinear set: \( C = \{q, \det(F, G) = 0\} \) corresponding to the set of points where \( F \) and \( G \) are collinear.

A singular control \( u_s \) is given by the relation:

\[\langle p[[F, G], F]\rangle + u_s(p[[F, G], G]) = 0.\]

Moreover a singular trajectory can be small time minimal or small time maximal. In the 2D-case, this status is tested by Lie brackets configurations as follows, see [4]. We introduce:

\[D = \det(G, [F, G], G), D'' = \det(G, G, F).\]

The trajectory is time-minimal if \( DD'' > 0 \) and time-maximal if \( DD'' < 0 \).

A straightforward computation gives in our case:

**Lemma 1.** The set \( S \) is given by: \( y[2(\Gamma - \gamma_+)z + \gamma_-] = 0 \) and if \( \gamma_+ \neq \Gamma \), the singular set is defined by the two lines \( y = 0 \) and \( z = \gamma_- / [2(\gamma_+ - \Gamma)] \).

The set \( C \) is defined by: \( \gamma_+z^2 + \Gamma y^2 - \gamma_-z = 0 \). It is a closed curve containing \((0, 0)\) and \((z_1, 0)\), with \( z_1 = \gamma_- / \gamma_+ \) (the equilibrium state of the free motion) which shrinks into \((0, 0)\) when \( \gamma_- = 0 \).

- If \( \gamma_- \neq 0 \), the intersection of \( C \) and \( S \) is empty except in the case where \( \gamma_+ = 2\Gamma \).

We represent on Fig. 1 the sets \( S \) and \( C \) for a situation with \( \gamma_- < 0 \) and \( \gamma_+ - \Gamma < 0 \).

**Lemma 2.** For the singular direction \( y = 0 \), we get:

\[DD'' = 2\gamma_+^2(\gamma_+ - \Gamma)(z - \gamma_+ / 2(\gamma_+ - \Gamma))(z - \gamma_+ / \gamma_+).

Near the origin, the sign is always positive if \( \gamma_- \neq 0 \). If \( \gamma_- = 0 \), the sign is given by \((\gamma_+ - \Gamma)\).

- For the singular direction \( z = \gamma_- / [2(\gamma_+ - \Gamma)] \), we have:

\[DD'' = \frac{y^2}{2(\gamma_+ - \Gamma)}[\gamma_+^2(\gamma_+ - 2\Gamma) - 4\Gamma y^2(\gamma_+ - \Gamma)^2].

Hence, near the origin one gets that \( DD'' > 0 \) if \( \gamma_+ - \Gamma < 0 \) and \( DD'' < 0 \) if \( \gamma_+ - \Gamma > 0 \).

One further step in the classification is to compute a normal form for the action of the feedback group. We have:

**Proposition 1.** The system is feedback equivalent to:

\[\begin{align*}
\dot{x} &= \frac{\gamma_-^2}{4(\Gamma - \gamma_+) - 2\Gamma x + (\Gamma - \gamma_+)z^2} \\
\dot{z} &= \frac{\gamma_-^2(\gamma_+ - 2\Gamma)}{2(\gamma_+ - \Gamma)} - \gamma_+ z + u_1
\end{align*}\]

**Proof:** Using polar coordinates:

\[y = \rho \cos \phi, \quad z = \rho \sin \phi,\]

one gets:

\[\dot{\rho} = \gamma_- \sin \phi + \rho(-\Gamma + (\Gamma - \gamma_+) \sin^2 \phi)\]

\[\dot{\phi} = \frac{-\rho \cos \phi}{\rho} + (\Gamma - \gamma_+) \sin(2\phi) + u_1.\]

If we use the coordinates \( x = \rho^2 / 2 \) and \( z \), the system becomes:

\[\dot{x} = -2\Gamma x + \gamma_- z + z^2(\Gamma - \gamma_+)\]

\[\dot{z} = \gamma_- - \gamma_+ z + u_1 \sqrt{2\rho - z^2}.

Making a feedback transformation of the form \( u_1 \rightarrow \beta u_1 \) where \( \beta \) is a function of \( x \) and \( z \), we can consider the system:

\[\dot{x} = -2\Gamma x + \gamma_- z + z^2(\Gamma - \gamma_+)\]

\[\dot{z} = \gamma_- - \gamma_+ z + u_1.\]
If we set \( z = Z + z_0 \) where \( z_0 = \gamma_−/[2(\gamma_+ − \Gamma)] \), we obtain the system in the form of the proposition.

In this simplified model, where the control is rescaled by the positive function \( \beta \), we keep most of the information about the initial system. In particular, all the feedback invariants in the plane minus the \( z \) axis are preserved: the collinear set corresponds to \( \dot{x} = 0 \) and the singular set is identified to \( z = 0 \) (and the optimality status is clear in the normal form).

For the simplified model, the adjoint system takes the form:

\[
\begin{align*}
\dot{p}_x &= 2\Gamma p_x \\
\dot{p}_z &= -2z \Gamma p_x (\Gamma - \gamma_+) + p_z \gamma_-
\end{align*}
\]
and can be easily integrated to compute the time-minimal synthesis with \( |u_1| \leq 1 \).

**B. The time-minimal syntheses**

We use [4] as general reference on time-minimal synthesis, see also [9]. The initial condition is fixed to \( q_0 = (0, 1) \) and we consider the problem of constructing the time-minimal synthesis from this initial point. This amounts to computing two objects:

- The switching locus \( \Sigma(q_0) \) of optimal trajectories which is deduced from the switching locus of extremal trajectories.
- The cut locus \( C(q_0) \) which is formed by the set of points where a minimizer ceases to be optimal.

In order to achieve this task, we must glue together local time minimal syntheses which are classified. To be more precise, take the case (d) of [18], the gluing being indicated on Fig. 2 on which we have represented the local extremal classifications of [4] which are crucial to deduce the optimal syntheses. In this case, the cut locus is a segment of the \( z \) axis starting at the initial point \( (0, 1) \) which is a consequence of a so-called elliptic situation. The switching locus is the union of a optimal singular trajectory, corresponding to a so-called hyperbolic point and a curve \( \Sigma_1(q_0) \) whose existence is related to a so-called parabolic situation. Note also the importance of the tangential point where arcs \( \sigma_+ \) and \( \sigma_- \) corresponding to \( u = +1 \) and \( u = -1 \) are tangent leading to the fish-shaped accessibility set \( A^+(q_0) \) represented on Fig. 3. This set is not closed since the arc \( \sigma^+ \) starting from \( (0, z_1) \) is not in \( A^+(q_0) \).

We next give the list of local syntheses we need to construct the global synthesis.

**List of local syntheses:**

We use the notation \( \sigma_1 \sigma_2 \) for an arc \( \sigma_1 \) followed by an arc \( \sigma_2 \). The first two cases are standard situations.

- **Ordinary switching points:** The local synthesis is given by \( \sigma_- \sigma_+ \) or \( \sigma_+ \sigma_- \). The two cases are distinguished using for instance the clock form \( \omega = pdq \) with \( \langle p, G \rangle = 0 \) and \( \langle p, F \rangle = 1 \) which is also useful to get more global results [4].
- **The fold case:** This case is due to the existence of singular directions located on \( S \). In this case the singular control is given by the relation:

\[
\langle p, [G, F], F \rangle + u_s \langle p, [G, F], G \rangle = 0.
\]

In order to be admissible, it must satisfy the constraint \( |u_s| \leq 1 \). Assume it is not saturating, i.e., \( |u_s| \neq 1 \) we have three cases:

1. **Hyperbolic case:** The singular arc is admissible and time minimal. The optimal synthesis is of the form \( \sigma_+ \sigma_+ \sigma_- \) where \( \sigma_+ \) is a singular arc.
2. **Elliptic case:** The singular arc is admissible but is small time maximal. An optimal arc is bang-bang with at most one switching. Not every extremal trajectory is optimal and we have existence of a cut locus.
3. **Parabolic point:** It corresponds to the existence of a singular arc for which \( |u_s| > 1 \). Every extremal curve is bang-bang with at most two switchings. In our case, the initial point is fixed and the switching locus starts with the intersection of \( \sigma_- \) with the singular line (see Fig. 4).

In our problem, two more complicated cases occur.

- **Saturating case:**
  
  A small time minimal singular trajectory is such that the singular control is saturating at a point \( M \). A consequence is the appearance of a switching curve at \( M \) (see Fig. 5).

- **A \( C \cap S \neq \emptyset \) case:**
  
  A time minimal singular trajectory meets the set \( C \) and becomes time maximal. (see Fig. 6)
The dynamics is linear and oscillating if the eigenvalues are complex or non oscillating if they are real. The switching function is defined as the angle between the solutions of the variational equation:

\[ p = -p \left( \frac{\partial F}{\partial q} + u \frac{\partial G}{\partial q} \right), \ u = \pm 1. \]

The corresponding dynamics is linear and \( p \) can be either oscillating if the eigenvalues are complex or non oscillating if they are real.

An equivalent but more geometric test is the use of the standard \( \Theta \)-function introduced in [9] and defined as follows. Let \( v \) be the tangent vector solution of the variational equation:

\[ \dot{\nu} = (\frac{\partial F}{\partial q} + u \frac{\partial G}{\partial q})v, \ u = \pm 1, \]

whose dynamics is similar to the one of the adjoint vector. Let \( 0, t \) be two consecutive switching times on an arc \( \sigma_+ \) or \( \sigma_- \). By definition, we have:

\[ p(0)G(q(0)) = p(t)G(q(t)) = 0. \]

We denote by \( v(\cdot) \) the solution of the variational equation such that \( v(t) = G(q(t)) \) and where this equation is integrated backwards from time \( t \) to time \( 0 \). By construction \( p(0)v(0) = 0 \) and we deduce that at time \( 0 \), \( p(0) \) is orthogonal to \( G(q(0)) \) and to \( v(0) \). Therefore, \( v(0) \) and \( G(q(0)) \) are collinear; \( \Theta(t) \) is defined as the angle between \( G(q(0)) \) and \( v(0) \) measured counterclockwise. One deduces that switching occurs when \( \Theta(t) = 0 \ [\pi] \). In the analytic case, \( \Theta(t) \) can be computed using Lie brackets. Indeed, for \( u = \varepsilon, \varepsilon = \pm 1 \), we have by definition

\[ v(0) = e^{-t ad(F + \varepsilon G)}G(q(t)), \]

and in the analytic case, the ad-formulae [4] gives:

\[ v(0) = \sum_{n \geq 0} \frac{(-t)^n}{n!} ad^n(F + \varepsilon G)G(q(t)). \]

In [18], the \( \Theta \)-function is computed using numerical simulations. Here, to make the computation explicit, we take advantage of the fact that we can lift our bilinear system into an invariant system onto the semi-direct product Lie group \( GL(2, \mathbb{R}) \times S \mathbb{R}^2 \) identified to the set of matrices of \( GL(3, \mathbb{R}) \):

\[ \begin{pmatrix} 1 & 0 & 0 \\ g & v \end{pmatrix}, \ g \in GL(2, \mathbb{R}), \ v \in \mathbb{R}^2, \]

acting on the subspace of vectors in \( \mathbb{R}^3 \):

\[ \begin{pmatrix} 1 \\ q \end{pmatrix}. \]

Lie brackets computations are defined as follows. We set:

\[ F(q) = Aq + a, \ G(q) = Bq, \]

and \( F, G \) are identified to \((A, a), (B, 0)\) in the Lie algebra \( gl(2, \mathbb{R}) \times \mathbb{R}^2 \). The Lie brackets computations on the semi-direct product Lie algebra are defined by:

\[ [A', a'], [B', b'] = ([A', B'], A'b' - B'a'). \]

We now compute \( \exp[-t ad(F + \varepsilon G)] \). The first step consists in determining \( \exp[-t ad(A + \varepsilon B)] \) which amounts to compute \( ad(A + \varepsilon B) \). We write \( gl(2, \mathbb{R}) = c \oplus sgl(2, \mathbb{R}) \) where \( c \) is the center

\[ \mathbb{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

We choose the following basis of \( sgl(2, \mathbb{R}) \):

\[ B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The matrix \( A \) is decomposed into:

\[ A = \begin{pmatrix} -\Gamma & 0 \\ 0 & -\gamma_+ \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2s \end{pmatrix}, \]

and hence \( \lambda = -\gamma_+ + \Gamma/2 \) and \( s = \gamma_+ - \Gamma/2 \). In the basis \((B, C, D)\), \( ad(A + \varepsilon B) \) is represented by the matrix:

\[ \begin{pmatrix} 0 & -2s & 0 \\ -2s & 0 & 2\varepsilon \\ 0 & -2\varepsilon & 0 \end{pmatrix}. \]

The characteristic polynomial is \( P(\lambda) = -\lambda(\lambda^2 + 4(\varepsilon^2 - s^2)) \) and the eigenvalues are \( \lambda = 0 \) and \( \lambda_i = \pm 2\sqrt{s^2 - \varepsilon^2}, i = 1, 2 \); \( \lambda_1 \) and \( \lambda_2 \) are distinct and real if \( |\gamma_+ - \Gamma| > 2 \) and we note
\[ \lambda_1 = 2\sqrt{s^2 - \varepsilon^2}, \lambda_2 = -\lambda_1; \lambda_1 \text{ and } \lambda_2 \text{ are distinct and imaginary if } |\gamma_+ - \Gamma| < 2 \text{ and we note } \lambda_1 = 2i\sqrt{s^2 - \varepsilon^2}, \lambda_2 = -\lambda_1. \text{ To compute } e^{-\text{tad}(A+\varepsilon B)}, \text{ we must distinguish two cases.} \\

\textbf{Real case:} \text{ In the basis } B, C, D, \text{ the eigenvectors corresponding to } \{0, \lambda_1, \lambda_2\} \text{ are respectively: } v_0 = (\varepsilon, 0, s), v_1 = (2s, -\lambda_1, 2\varepsilon) \text{ and } v_2 = (2s, -\lambda_2, 2\varepsilon). \text{ Therefore, in this eigenvector basis, } \exp[-\text{tad}(A+\varepsilon B)] = \text{ the diagonal matrix: } \text{diag}(1, e^{-\lambda_1 t}, e^{-\lambda_2 t}). \text{ To compute } \exp[-\text{tad}(A+\varepsilon B)]B, \text{ we use the decomposition } B = \alpha v_0 + \beta v_1 + \beta v_2, \text{ with:} \\
\alpha = \frac{s}{\varepsilon_1 - s^2}, \beta = \frac{s}{4(s_1^2 - \varepsilon_2^2)}. \text{ Hence one gets:} \\
\[ e^{-\text{tad}(A+\varepsilon B)}B = \alpha v_0 + \beta e^{-\lambda_1 t} v_1 + \beta e^{-\lambda_2 t} v_2. \]
To test the collinearity at \( q_0 \), we compute \\
\[ \text{det}(B(q_0), e^{-\text{tad}(A+\varepsilon B)}B(q_0)) = 0 \]
where the determinant is equal to \\
\[ (z_0^2 - y_0^2)\alpha s + 2\beta \varepsilon (\beta e^{-\lambda_1 t} + \beta e^{-\lambda_2 t}) + 2\gamma y_0 z_0 (\lambda_1 + \lambda_2) \]
which simplifies into \\
\[ (z_0^2 - y_0^2)\alpha s + 4\beta \varepsilon (\lambda_1 + \lambda_2) \]

\textbf{Imaginary case:} \text{ In this case, we note } \lambda_1 = i\varepsilon \text{ the eigenvalue associated to the eigenvector } (2s, -i\varepsilon, 2\varepsilon). \text{ We consider the real part } v_1 = (2s, 0, 2\varepsilon) \text{ and the imaginary part } v_2 = (0, -\varphi, 0). \text{ In the basis } v_0 = (\varepsilon, 0, s), v_1, v_2, \text{ ad}(A+\varepsilon B) \text{ takes the normal form:} \\
\[ \text{diag}(0, \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix}). \]
Hence, we have in this basis: \\
\[ e^{-\text{tad}(A+\varepsilon B)} = \text{diag}(1, \begin{pmatrix} \cos(\varphi t) & -\sin(\varphi t) \\ \sin(\varphi t) & \cos(\varphi t) \end{pmatrix}). \]
We decompose \( B \) in the same basis: \( B = \alpha v_0 + \beta v_1 + \gamma v_2 \), where \\
\[ \alpha = \frac{\varepsilon}{\varepsilon_1 - s^2}, \beta = \frac{s}{2(s_1^2 - \varepsilon_2^2)}, \gamma = 0. \]
Hence, we get: \\
\[ e^{-\text{tad}(A+\varepsilon B)}B = \alpha v_0 + \beta \cos(\varphi t) v_1 + \gamma \sin(\varphi t) v_2. \]
Computing we obtain: \\
\[ \text{det}(B(q_0), e^{-\text{tad}(A+\varepsilon B)}B(q_0)) = (z_0^2 - y_0^2)(\alpha s + 2\beta \varepsilon \cos(\varphi t) + 2\beta \varphi \sin(\varphi t) y_0 z_0) = 0 \]

\textbf{Proposition 2.} \text{ Assume } \gamma_- = 0 \text{ and that a switching occurs at times } 0, \text{ along an arc } \sigma_\varepsilon \text{ initiating from } (y_0, z_0). \text{ Then} \\
1) \text{ if } |\gamma_+ - \Gamma| > 2, \text{ we must have:} \\
\[ (z_0^2 - y_0^2)(\alpha s + 4\beta \varepsilon \cos(\lambda_1 t)) - 4\gamma y_0 z_0 \beta \lambda_1 \sinh(\lambda_1 t) = 0 \]
where \( \lambda_1 = 2\sqrt{s^2 - \varepsilon^2}, \alpha = \frac{\varepsilon}{s_1 - s_2}, \beta = \frac{s}{4(s_1^2 - s_2^2)}. \)
In particular if \( (y_0, z_0) = (0, 1) \) there is no switching for \( t > 0. \)
2) \text{ if } |\gamma_+ - \Gamma| < 2, \text{ we must have:} \\
\[ (z_0^2 - y_0^2)(\alpha s + 2\beta \varepsilon \cos(\varphi t) + 2\beta \varphi \sin(\varphi t) y_0 z_0 = 0 \]
where \( \varphi = 2\sqrt{s^2 - \varepsilon^2}, \alpha = \frac{\varepsilon}{s_1 - s_2}, \beta = \frac{s}{4(s_1^2 - s_2^2)}. \)
In particular if \( (y_0, z_0) = (0, 1) \) switching occurs periodically with a period \( 2\pi/\varphi. \)

\textbf{Case } \gamma_- \neq 0: \text{ The computations are more complex but this case is similar. The vector field } F + \varepsilon G \text{ is an affine vector field and to simplify the computations it is transformed into the linear vector field } A + \varepsilon B \text{ making the following translation in the } \mathbb{R}^2 \text{ space: } Y = y + y, Z = z + \bar{z} \text{ with } y = \varepsilon \gamma_-/(\Gamma + \varepsilon^2) \text{ and } \bar{z} = -\gamma_-/(\Gamma + \varepsilon^2). \text{ G is transformed into the affine vector } Bq + w \text{ where } w \text{ is the vector } (w_1, w_2) \text{ with } w_1 = -\gamma_-/(\Gamma + \varepsilon^2) \text{ and } w_2 = -\gamma_-/(\Gamma + \varepsilon^2). \text{ The vector field } \text{ad}(F + \varepsilon G) \text{ acts on the vector space } gl(2, \mathbb{R}) \oplus \mathbb{R}^2 \text{ and the action on the space } gl(2, \mathbb{R}) \text{ has been previously computed. According to the definition of the Lie bracket, the action on the } \mathbb{R}^2 \text{ space is simply the action of the linear operator } A + \varepsilon B. \text{ The characteristic polynomial is } P = \lambda^2 + (\Gamma + \gamma_+) \lambda \text{ and } (\Gamma + \gamma_+ + \varepsilon^2). \text{ We must distinguish two cases:} \\
\textbf{Real case:} \text{ If } |\Gamma - \gamma_+| > 2, \text{ we have two real eigenvalues} \\
\[ \sigma_1 = -(\Gamma + \gamma_+) + \sqrt{\sigma^2 - \varepsilon^2}/2, \sigma_2 = -(\Gamma + \gamma_+) - \sqrt{\sigma^2 - \varepsilon^2}/2 \]
with corresponding eigenvectors \( f_1 \) and \( f_2 \). Writing the vector \( w \) as \( \delta_1 f_1 + \delta_2 f_2 \), one gets using the previous computations \\
\[ e^{-\text{tad}(F + \varepsilon G)}G = \alpha v_0 + \beta \varepsilon \lambda_1 \beta v_1 + \gamma e^{-\lambda_2 t} v_2 + \delta_1 e^{-\sigma_1 t} f_1 + \delta_2 e^{-\sigma_2 t} f_2. \]

\textbf{Complex case:} \text{ If } |\Gamma - \gamma_+| < 2, \text{ we have two complex eigenvalues } -(\Gamma + \gamma_+) \pm 2\sqrt{\sigma^2 - \varepsilon^2}/2. \text{ The computation is similar using a real Jordan normal form for the exponential of the operator.} \\

\textbf{Generalization:} \text{ This technique can be generalized to the time-minimal control problem in the full control case, replacing the control domain } |u| \leq 1 \text{ by } |u_1|, |u_2| \leq 1. \\

\textbf{D. Classification of the optimal syntheses} \\
We describe the different time-optimal syntheses in the single-input case. Without loss of generality, we restrict the study to the initial point \( q_0 = (0, 1) \). The classification is done with respect to the relative positions of the feedback invariants \( C \) and \( S \) to the optimal status of singular extremals which are time minimal or time maximal according to the values of \( \gamma_+ \) and \( \gamma_- \). \\
For \( \gamma_- = 0 \), the set \( C \) is restricted to the origin and we have two cases according to the sign of \( \Gamma - \gamma_- \). Note that the form of the extremals \( \sigma_+ \) and \( \sigma_- \) starting from \( q_0 \) depends on the sign of \( |\Gamma - \gamma_+| - 2 \). Two cases for \( \Gamma > \gamma_+ \) are presented in [18]. We complete this study with the optimal synthesis for \( \Gamma < \gamma_+ \) and \( |\Gamma - \gamma_+| < 2 \) displayed in Fig. 7a. \\
For \( \gamma_- \neq 0 \), we distinguish four cases according to the signs of \( \gamma_- \) and \( \Gamma - \gamma_- \). One case (\( \Gamma > \gamma_+ \) and \( \gamma_- < 0 \)) is treated in [18]. We consider here three types of optimal synthesis represented in Fig. 7b, 7c and 7d. Note that in a same class of synthesis the reachable set from the initial point
\( q_0 \) depends on the dissipative parameters which can modify the structure of the synthesis. The last case \( \gamma_- > 0 \) and \( \gamma_+ > \Gamma \) can be deduced from the case \( \gamma_- > 0 \) and \( \gamma_+ < \Gamma \) since the horizontal singular line plays no role in both cases. The synthesis of Fig. 7d is very similar to the one of Fig. 2 except the fact that a part of the horizontal singular line is admissible. The switching locus has been computed numerically using the switching function \( \Phi \).

The role of the parameter \( \gamma_- \) is clearly illustrated in Figs. 7a and 7c. The case \( \gamma_- = 0 \) is a degenerate case where the set \( C \) shrinks into a point. The variation of \( \gamma_- \) induces a bifurcation of the control system leading to new structures of the optimal synthesis. For \( \gamma_- \neq 0 \), the set \( C \) is a non trivial closed curve. The optimal status of the vertical singular line changes when this line crosses the set \( C \) in Fig. 7c.

III. The B1-Input Case

A. Geometric analysis

The system is written in short in Cartesian coordinates as follows:

\[
\dot{q} = F_0(q) + u_1 F_1(q) + u_2 F_2(q), \quad |u| \leq 1.
\]

Introducing the Hamiltonians \( H_i = \langle p, F_i \rangle, \quad i = 0, 1, 2 \), the pseudo-Hamiltonian associated to the time-optimal control problem is:

\[
H = H_0 + \sum_{i=1}^{2} u_i H_i + p_0,
\]

where \( p_0 \leq 0 \). The time-optimal control is given outside the switching surface \( \Sigma: H_1 = H_2 = 0 \), by \( u_i = H_i/\sqrt{H_i^2 + H_2^2} \), \( i = 1, 2 \), with the corresponding true Hamiltonian:

\[
H_r = H_0 + \sqrt{H_1^2 + H_2^2},
\]

whose solutions (outside \( \Sigma \)) are smooth and are called extremals of order 0. More general non smooth extremals can be obtained by connecting such arcs through \( \Sigma \).

To make the geometric analysis and to highlight the symmetry of revolution, the system is written using the spherical coordinates:

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi
\]

and a feedback transformation:

\[
v_1 = u_1 \cos \theta + u_2 \sin \theta, \quad v_2 = -u_1 \sin \theta + u_2 \cos \theta.
\]

We obtain the system:

\[
\begin{align*}
\dot{\rho} & = \gamma_- \rho \sin \phi - \rho (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi) \\
\dot{\phi} & = -\frac{\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2} (\gamma_+ - \Gamma) + v_2 \\
\dot{\theta} & = -\cot \phi v_1.
\end{align*}
\]

Hence, one deduces that the true Hamiltonian is:

\[
H_r = |\gamma_- \rho \sin \phi - \rho (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)| p_\rho + \rho [\frac{-\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2} (\gamma_+ - \Gamma)] + \sqrt{p_\rho^2 + p_\phi^2 \cot^2 \phi}.
\]

From this, we deduce the following lemma:
Lemma 3. (i) The angle $\theta$ is a cyclic variable and $p_\theta$ is a first integral (symmetry of revolution).
(ii) For $\gamma_- = 0$, using the coordinate $r = \ln \rho$, the Hamiltonian takes the form:
\[
H_r = -\left(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi\right)p_r + \sin(2\phi)(\gamma_+ - \Gamma)p_\phi + \sqrt{p_\rho^2 + p_\phi^2 \cot^2 \phi}.
\]
Hence, $r$ is an additional cyclic variable and $p_r$ is a first integral. The system is thus Liouville integrable.

As a consequence, we can deduce two properties. First of all, the $z$-axis is an axis of revolution and the state $q_0 = (0, 0, 1)$ is a pole. This means that by making a rotation around $(Oz)$ of the extremal synthesis for the 2D-system, we generate the extremal synthesis for the 3D-system.

More generally, we have for $\gamma_- = 0$ a system on the two-sphere of revolution described by Eqs. (6b) and (6c) coupled with the one dimensional system (6a) describing the evolution of the physical variable $\rho$ corresponding to the purity of the system. Moreover, the system is invariant for the transformation $\phi \mapsto \pi - \phi$ which is associated to a reflexional symmetry with respect to the equator for the system (6) restricted to the two-sphere of revolution. This property is crucial in the analysis of the integrable case.

If $\gamma_- \neq 0$ then the situation is more intricate. The extremals solutions of order 0 satisfy the equations which are singular for $\rho = 0$:
\[
\begin{align*}
\dot{r} &= \gamma_- \cos \phi - \rho(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi), \\
\dot{\phi} &= -\frac{\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2} (\gamma_+ - \Gamma) + \frac{p_\phi}{Q}, \\
\dot{\theta} &= \frac{p_\theta \cot^2 \phi}{Q},
\end{align*}
\]
and
\[
\begin{align*}
\dot{p}_r &= (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r - \frac{\gamma_- \sin \phi}{\rho^2} p_\phi, \\
\dot{p}_\phi &= \left[\gamma_- \sin \phi + \rho (\Gamma - \gamma_+) \sin(2\phi)\right]p_r - \frac{1}{\rho} \cos \phi \gamma_- - (\gamma_+ - \Gamma) \cos(2\phi) p_\phi + \frac{p_\phi^2 \cos \phi}{Q \sin^2 \phi}, \\
\dot{p}_\theta &= 0,
\end{align*}
\]
where $Q = \sqrt{p_\rho^2 \cot^2 \phi + p_\phi^2}$.

B. Regularity analysis

The smooth extremal curves solutions of $\tilde{H}_r$ are not the only extremals because more complicated behaviors are due to the existence of the switching surface $\Sigma$: $H_1 = H_2 = 0$. Hence, in order to get singularity results, we must analyze the possible connections of two smooth extremals crossing $\Sigma$ to generate a piecewise smooth extremal. This can also generate complex singularities of the Fuller type, where the switching times accumulate. In our problem, the situation is less complex because of the symmetry of revolution. The aim of this section is to make the singularity analysis of the extremals near $\Sigma$.

The structure of optimal trajectories is described by the following proposition.

Proposition 3. Every optimal trajectory is:

- Either an extremal trajectory with $p_\theta = 0$ contained in a meridian plane and time-optimal solution of the 2D-system, where $u = (u_1, 0)$.
- Or subarcs solutions of $\tilde{H}_r$, where $p_\theta \neq 0$ with possible connections in the equator plane for which $\phi = \pi/2$.

Proof: The first assertion is clear. If $p_\theta = 0$ then extremals are such that $\dot{\theta} = 0$ and up to a rotation around the $z$-axis, they correspond to solutions of the 2D-system. The switching surface $\Sigma$ is defined by: $p_\rho \cot \phi = \phi = 0$. We cannot connect an extremal with $p_\theta \neq 0$ to an extremal where $p_\theta = 0$ since at the connection the adjoint vector has to be continuous. Hence, the only remaining possibility is to connect subarcs of $\tilde{H}_r$, with $p_\theta \neq 0$ at a point of $\Sigma$ leading to the conditions $p_\theta = 0$ and $\phi = \pi/2$.

Further work is necessary to analyze the behaviors of such extremals near $\Sigma$.

Normal form: A first step in the analysis is to construct a normal form. Taking the system in spherical coordinates and setting $\psi = \pi/2 - \phi$, the approximation is:
\[
\begin{align*}
\dot{\rho} &= \gamma_- \psi - \rho [\Gamma + (\gamma_+ - \Gamma) \psi^2] \\
\dot{\psi} &= \frac{\gamma_+}{\rho} (1 - \psi^2/2) - \psi (\gamma_+ - \Gamma) - v_2 \\
\dot{v}_2 &= -\psi v_1,
\end{align*}
\]
with the corresponding Hamiltonian:
\[
H_r = p_\rho [\gamma_- \psi - \rho (\Gamma + (\gamma_+ - \Gamma) \psi^2)] + p_\psi \left[\frac{\gamma_+}{\rho} (1 - \psi^2/2) - \psi (\gamma_+ - \Gamma)\right] + \sqrt{p_\rho^2 + p_\psi^2 \psi^2}.
\]

Proposition 4. Near $\psi = 0$, $p_\psi = 0$, we have two distinct cases for optimal trajectories:

- If $\gamma_- = 0$, for the 2D-system, the line $\psi = 0$ is a singular trajectory with admissible zero control if $\gamma_- \neq 0$. It is time maximal if $(\gamma_+ - \Gamma) > 0$ and time minimal if $(\gamma_+ - \Gamma) < 0$. Hence, for this system, we get only extremal trajectories through $\Sigma$ in the case $(\gamma_+ - \Gamma) < 0$, where $\psi$ is a solution to a first order $t^2$. They are the only non-smooth optimal trajectories passing through $\Sigma$.

- If $\gamma_- \neq 0$, for the 2D-system, the line $\psi = p_\psi = 0$ becomes a set of ordinary switching points where $\psi$ and $p_\psi$ are of order $t$. Moreover, connections for extremals of $\tilde{H}_r$ are eventually possible, depending upon the set of parameters and initial conditions.

Proof: For the normal form, the adjoint system is:
\[
\begin{align*}
\dot{p}_\rho &= p_\rho (\Gamma + (\gamma_+ - \Gamma) \psi^2) + \frac{p_\psi}{\rho} (1 - \psi^2/2) \\
\dot{p}_\psi &= -p_\rho (\gamma_- - 2\psi \rho (\gamma_+ - \Gamma)) + p_\psi \left[\frac{\gamma_+}{\rho} (1 - \psi^2/2) - (\gamma_+ - \Gamma)\right] + v_1 p_\rho.
\end{align*}
\]
In order to make the evaluation of smooth arcs reaching or departing from $\Sigma$, the technique is simple: a solution of the form $\psi(t) = at + o(t)$, $p_\psi(t) = bt + o(t)$ is plugged in the equations to determine the coefficients. From the equations,
we observe that the contacts with $\Sigma$ differ in the case $\gamma_- = 0$ from the case $\gamma_\neq 0$ that we discuss separately.

First of all, we consider the case $\gamma_- = 0$; $p_\theta = 0$, $\psi = 0$ is an admissible singular direction (with zero control) which can be slow if $(\gamma_+ - \Gamma) > 0$ or fast if $(\gamma_+ - \Gamma) < 0$. In the first case, there is no admissible extremal through $\Sigma$ while it is possible if $\gamma_+ - \Gamma < 0$. If we compute the different orders, we have that $\psi$ is of order $t$, $p_\psi$ is of order $t^2$ while $p_\rho$ has to be non zero if $p_\theta = 0$. If we consider extremals with $p_\theta \neq 0$, we can conclude with the orders alone. Indeed the Hamiltonian is $H_r = \varepsilon$, $\varepsilon = 0, 1$ and in both cases, we have:

$$-p_\rho\rho(\gamma_+ - \Gamma)\psi^2 - p_\psi\psi(\gamma_+ - \Gamma) + \sqrt{p_\rho^2 + p_\psi^2} = 0.$$  

The conclusion using orders is then straightforward. For instance, if $\psi$ and $p_\psi$ are of order one, this gives $p_\psi = p_\theta = 0$ which is impossible. The other cases are similar.

In the case $\gamma_- \neq 0$, the analysis is more intricate and we must analyze the equations. We introduce the Hamiltonians:

$$H_1 = -p_\theta\psi, \quad H_2 = p_\rho.$$  

Differentiating $H_1$ and $H_2$ with respect to $t$, one gets:

$$\dot{H}_1 = \{H_1, H_0\} + v_2 H_1, \quad H_2 = \{H_2, H_0\} + v_1 H_2$$

and at a point of $\Sigma$, we obtain the relations:

$$\dot{H}_1 = -p_\theta(\gamma_- - v_2), \quad \dot{H}_2 = \gamma_- p_\rho - v_1 p_\theta.$$  

In order to analyze the singularity, we use a polar blowing up:

$$H_1 = r \cos \alpha, \quad H_2 = r \sin \alpha,$$

and we get:

$$\dot{r} = \gamma_-\left[- \frac{p_\theta \cos \alpha}{\rho} + p_\rho \sin \alpha\right],$$

$$\dot{\alpha} = \frac{1}{2}[\gamma_- p_\rho \cos \alpha + \frac{p_\theta \gamma_- \sin \alpha}{\rho} - p_\theta].$$

Hence, the extremals crossing $\Sigma$ are given by solving $\dot{\alpha} = 0$, while the sign of $\dot{r}$ is given by the first equation above.

Depending upon the parameters and the initial conditions on $(p_\rho, \rho)$, the equation $\dot{\alpha} = 0$ can have at most two distinct solutions on $(0, 2\pi)$, while in the case $p_\theta = 0$, we get an ordinary switching point for the single-input system. The assertion 4 is proved.

C. Geometric analysis and numerical solution

We first analyze the integrable case $\gamma_- = 0$. We only present a summary of the result of [5] in order to be generalized to the case $\gamma_\neq 0$.

1) The case $\gamma_- = 0$: The system (7) is associated to a system on the two-sphere of revolution of the form:

$$\dot{q} = G_0(q) + \sum_{i=1}^{2} u_i G_i(q).$$

It defines a Zermelo navigation problem [2] on the two-sphere of revolution where the drift term $G_0$ represents the current:

$$G_0 = \frac{\sin(2\phi)}{2}(\gamma_+ - \Gamma) \frac{\partial}{\partial \phi},$$

and $G_1 = \frac{\partial}{\partial \phi}$, $G_2 = -\cot \phi \frac{\partial}{\partial \phi}$ form a frame for the metric $g = d\phi^2 + \tan^2 \phi d\theta^2$ which is singular at the equator $\phi = \pi/2$. The drift can be compensated by a feedback with $|u| < 1$ if $|\gamma_+ - \Gamma| < 2$. This leads to the following discussion.

Case $|\gamma_+ - \Gamma| < 2$: In this case, the system reduced to the two-sphere defines a Finsler geometry for which the extremals are a deformation of the extremals of $g = d\phi^2 + \tan^2 \phi d\theta^2$. The main problem properties [11] are described in the next proposition.

Proposition 5. If for fixed $(p_\rho, p_\theta)$, the level set of $H_r = \varepsilon$ ($\varepsilon = 0, 1$) is compact without singular point and has a central symmetry with respect to $(\phi = \pi/2, p_\phi = 0)$ then it contains a periodic trajectory $(\phi, p_\phi)$ of period $T$ and if $p_\phi^0(0)$ are distinct, we have two distinct extremal curves $q^+(t)$, $q^-(t)$ starting from the same point and intersecting with the same length $T/2$ at a point such that $\phi(T/2) = \pi - \phi(0)$ (see Fig. 8).

Case $|\gamma_+ - \Gamma| > 2$: We have two types of extremals characterized by their projection on the two-sphere: those occurring in a band near the equator and described by proposition 5 and those crossing a band near $\phi = \pi/4$ and with asymptotic properties of proposition 6:

Proposition 6. If $|\Gamma - \gamma_+| \geq 2$ then we have extremal trajectories such that $\phi \to 0$, $|p_\phi| \to +\infty$ when $t \to +\infty$ while $\theta \to 0$.

Both behaviors are represented on Fig. 9.

2) The case $\gamma_- \neq 0$: We present numerical results about the behavior of extremal solutions of order 0 and conjugate point analysis.

Extremal trajectories:

We begin by analyzing the structure of extremal trajectories. The description is based on a direct integration of the system (8). We observe two different asymptotic behaviors corresponding to stationary points of the dynamics which are
described by the following results.

**Proposition 7.** In the case denoted (a) where \( |p_\phi(t)| \to +\infty \) when \( t \to +\infty \), the asymptotic stationary points \((\rho_f, \phi_f, \theta_f)\) of the dynamics are given by \( \rho_f = |\gamma_-|/\sqrt{\Gamma + \Gamma^2} \) and \( \phi_f = \arctan(1/\Gamma) \) if \( \gamma_- > 0 \) or \( \phi_f = \pi - \arctan(1/\Gamma) \) if \( \gamma_- < 0 \).

**Proof:** We assume that \( |p_\phi(t)| \to +\infty \) as \( t \to +\infty \) and that \( \cot(\phi) \) remains finite in this limit. One deduces from the system (8) that \((\rho_f, \phi_f)\) satisfy the following equations:

\[
\gamma_- \cos \phi_f = \rho_f (\gamma_+ \cos^2 \phi_f + \Gamma \sin^2 \phi_f)
\]

\[
\gamma_- \sin \phi_f = (\gamma_+ - \Gamma) \cos \phi_f \sin \phi_f + \varepsilon,
\]

where \( \varepsilon = \pm 1 \) according to the sign of \( p_\phi \). The quotient of the two equations leads to

\[
(\gamma_+ - \Gamma) \cos \phi_f + \varepsilon = \tan \phi_f (\gamma_+ \cos^2 \phi_f + \Gamma \sin^2 \phi_f)
\]

which simplifies into

\[
\tan \phi_f = \frac{\varepsilon}{\Gamma}.
\]

Using the fact that \( \phi_f \in [0, \pi] \) and \( \gamma_- \cos \phi_f \geq 0 \), one arrives to \( \phi_f = \arctan(1/\Gamma) \) if \( \gamma_- > 0 \) and \( \phi_f = \pi - \arctan(1/\Gamma) \) if \( \gamma_- < 0 \). From the equation

\[
\gamma_- \cos \phi_f = \rho_f (\gamma_+ \cos^2 \phi_f + \Gamma \sin^2 \phi_f),
\]

one finally obtains that

\[
\rho_f = \frac{\gamma_- \sqrt{1 + \Gamma^2}}{1 + \gamma_+ \Gamma}.
\]

**Proposition 8.** In the case denoted (b) where \( \lim_{t \to +\infty} \phi(t) = 0 \) or \( \pi \), the asymptotic limit of the dynamics is characterized by \( \rho_f = |\gamma_-|/\gamma_+ \) and \( \phi_f = 0 \) if \( \gamma_- > 0 \) or \( \phi_f = \pi \) if \( \gamma_- < 0 \).

**Proof:** Using the relation

\[
\gamma_- \cos \phi_f = \rho_f (\gamma_+ \cos^2 \phi_f + \Gamma \sin^2 \phi_f),
\]

one deduces that \( \gamma_- \cos \phi_f \geq 0 \) and that \( \rho_f = |\gamma_-|/\gamma_+ \) if \( \phi_f = 0 \) or \( \pi \).

We have numerically checked that if \( |\Gamma - \gamma_+| > 2 \) then only the case (a) is encountered whereas if \( |\Gamma - \gamma_+| < 2 \), the extremals are described by cases (a) and (b). One particularity of the case (a) is the fact that the limit of the dynamics only depends on \( \Gamma \) and on the sign of \( \gamma_- \) and not on \( \phi(0) \) or \( \gamma_+ \). The structure of the extremals is also simple in case (b) since the limit of \( \phi \) is 0 or \( \pi \) independently of the values of \( \Gamma \), \( \gamma_+ \) or \( \gamma_- \). The different behaviors of the extremals are illustrated in Fig. 10 for the case \( |\Gamma - \gamma_+| > 2 \) and in Fig. 12 for the case \( |\Gamma - \gamma_+| < 2 \). The corresponding optimal control fields \( v_1 \) and \( v_2 \) are represented in Fig. 11 for the case (a) and in Fig. 13 for the case (b). In Fig. 11, note that the control \( v_1 \) tends to 0 whereas \( v_2 \) is close to -1 for \( t \) sufficiently large. This is due to the fact that \( |p_\phi| \to +\infty \) when \( t \to +\infty \) and can be easily checked from the definition of \( v_1 \) and \( v_2 \). We observe a similar behavior for the case (b) in Fig. 13. The control field \( v_2 \) acquires here a bang-bang structure which is related to the unbounded and oscillatory behavior of \( p_\phi(t) \) (see Fig. 13).

**Conjugate points:**

The Cotcot code is used to evaluate the conjugate points. This occurs only in case (b) and the numerical simulations give that the first conjugate points appear before an uniform number of oscillations of the \( \phi \) variable. This phenomenon is represented on Fig. 14. Cutting the trajectory at the first conjugate point avoids such a behavior. Note that due to the symmetry of revolution, the global optimality is lost for \( \theta \leq \pi \).

**IV. PHYSICAL CONCLUSIONS**

We give some qualitative conclusions on the time-optimal control of two-level dissipative systems. The discussion concerns the role of dissipation which can be beneficial or not for the dynamics and the robustness with respect to dissipative parameters of the optimal control.

The dissipation effect is well summarized by Fig. 2. In this case \( \Gamma > \gamma_+ \) and one sees that as long as the purity of the state decreases (for \( 0 \leq z \leq 1 \)), it is advantageous to use a control field, the dissipation being undesirable. On the contrary, when the purity starts increasing (for \( -\gamma_- \leq z \leq 0 \)) then the dissipation alone becomes more efficient and its role positive. The quickest way to accelerate the purification of the state consists in letting the dissipation act. This constitutes a non-intuitive physical conclusion which, however, crucially depends on the respective values of \( \Gamma \) and \( \gamma_+ \). For instance, if \( \gamma_+ > \Gamma \) then all the preceding conclusions are modified. This result can also be interpreted in the control of a spin 1/2 particle by magnetic fields. Equation (5) describes the evolution of the spin system in the basis of the \( z \) component of the spin, the magnetic field being only along the \( z \)– and \( y \)– axis. The north and the south poles of the Bloch sphere correspond to the two positions of the spin along the \( z \)-axis. An interesting question is to change in minimum time the orientation of the spin system i.e. to pass from the north pole to the south pole. In the conservative case, the solution is
Fig. 10. Extremal trajectories for $\Gamma = 4.5$, $\gamma_+ = 2$ and $\gamma_- = -0.5$. The equations of the dashed lines are $\phi = \pi - \arctan(1/\Gamma)$ and $\rho = |\gamma_-|\sqrt{1 + \Gamma^2}/(1 + \gamma_+ \Gamma)$ (see the text). Numerical values of the parameters are taken to be $\phi(0) = \pi/4$, $p_{\phi}(0) = 0.1$ and $\rho(0) = 1$. $p_{\phi}(0)$ is successively equal to -10, -2.5, 0, 2.5 and 10 for the different extremals.

Fig. 11. Plot of the optimal control fields $v_1$ (solid line) and $v_2$ (dashed line) as a function of time $t$ for the extremal trajectory of Fig. 10 with $p_{\phi}(0) = 5$. The equation of the horizontal solid line is $v = 0$.

Fig. 12. Same as Fig. 10 but for $\Gamma = 2.5$. The equation of the dashed line is $\rho = |\gamma_-|/\gamma_+$. are due to spontaneous emission. In this case, the answer to the optimal control problem is given by Figure 7d. The solution is a combination of a bang arc and a singular arc along the $z-$axis with a control field equal to 0. The south pole is reached asymptotically in infinite time. Other optimal syntheses of Figure 7 can be interpreted along the same lines.

The robustness of the optimal control with respect to dissipative parameters is illustrated by the double-input control. We give different examples. If $\gamma_- = 0$ then the integrability of the Hamiltonian and the geometrical properties of the extremals are preserved when $|\Gamma - \gamma_+| < 2$. If $\gamma_- \neq 0$ then the asymptotic behavior of the extremals slightly depends on the parameters $\Gamma$, $\gamma_+$ and $\gamma_-$ (see Propositions 7 and 8). Fig. 11 and 13 show that the extremal control fields have also asymptotic behaviors independent of the dissipation. In case (a), the control fields tend to a constant whereas a bang-bang structure appears in case (b). This conclusion could be interesting for practical applications where robustness with respect to physical parameters and simple control fields are needed. In addition, due to the simple structure of the time-optimal synthesis, shooting techniques will be particularly efficient to determine the control fields especially in case (a).

ACKNOWLEDGMENT

We acknowledge support from the Agence Nationale de la recherche (ANR CoMoc).
the horizontal and vertical solid lines are respectively $p = 3.386$ and $3.535$ for coordinates $\theta$.

Fig. 13. (top) Same as Fig. 11 but for the extremal of Fig. 12 with $p_0(0) = 2.5$. (bottom) Evolution of $p_0$ for the same extremal as a function of $t$.

Fig. 14. Plot of the extremals of Fig. 12 up to the first conjugate point. The coordinates $\theta$ of the conjugate points are respectively $3.149, 3.116, 3.332, 3.386$ and $3.535$ for $p_0(0)$ equal to $-10, -2.5, 0, 2.5$ and $10$. The equations of the horizontal and vertical solid lines are respectively $\phi = \pi/2$ and $\theta = \pi$.

REFERENCES