SINGULAR TRAJECTORIES AND DECOUPLING VECTOR FIELDS FOR UNDERWATER VEHICLES

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Abstract: We are concerned with the class of mechanical systems of the form kinetic minus potential energy. This paper is a first attempt to analyze the relation between singular trajectories for the time optimal problem and the existence of decoupling vector fields. We focus on a very specific application: underwater vehicles. We first derive the conditions for an extremal to be singular for the time optimal problem. Then, we show that the order of the singular extremals is non generic, i.e. is at least 2. Related to this property is the existence of vanishing vector fields along such extremals. Finally, we show that these vanishing vector fields are decoupling vector fields for the system. We draw the consequences for the trajectory design problem.

Keywords: Underwater Vehicles, Singular Trajectories, Decoupling Vector Fields

1. INTRODUCTION

A very important class of control systems, even though they are non-generic, is the class of controlled mechanical systems. Examples of these systems include the planar rigid body with a single variable direction thruster, the snakeboard, and underwater vehicles; see for instance Bullo et al. (2000); Chyba et al. (2003); Lewis et al. (1994). In this paper, we focus on a class of underwater vehicles. Currently, the use of underwater vehicles is expanding in every area of oceanography and during the past few years, much research has gone into building autonomous underwater vehicles to perform tasks for which human divers and small human-driven submarines are not qualified. These autonomous robots are confronted to the task of planning their own motions. This task is a crutial one for the vehicle, and has yet to be fully investigated. In light of our recent results in Chyba and Haberkorn (2005) on singular extremals for the time minimal problem for a simplified model of underwater vehicle, we propose to analyze the connection between the subjects of kinematic reduction for mechanical systems with no potential or external forces, see Bullo (2004); Bullo et al. (2002); Bullo and Lewis (2004), and the one on singular trajectories. The interest with considering singular trajectories to solve the motion planning problem is that they are solution of the maximum principle, see Bonnard and Chyba (2003), and then candidates for optimality. Moreover, they are related to simple motion of the systems such as pure translations and rotations. In this paper we explicit the existence of decoupling vector fields in the simplified situation and discuss a possible generalisation for a model with potential and external forces.

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2. EQUATIONS OF MOTION FOR UNDERWATER VEHICLES

Underwater vehicles belong to the category of mechanical systems derived from a Lagrangian of the form kinetic energy minus potential energy. Let $\mathcal{Q}$ be the configuration manifold, r be the number of degrees of freedom of the system, and $q = (q_1, \cdots, q_r)$ local coordinates on $\mathcal{Q}$. In the local coordinates $(q, \dot{q})$ on $T\mathcal{Q}$, the Lagrangian L takes the form:

$$L_q(q, \dot{q}) = \frac{1}{2}\dot{q}^t M_q(q)\dot{q} - V_q(q)\,, \tag{1}$$

where $M_q(q)$ is a symmetric positive definite $r \times r$ matrix and $V_q(q)$ is the potential energy function. The Euler-Lagrange equations provides the equations of motion for the unforced system: $M_q(q)\ddot{q} + C_q(q, \dot{q})\dot{q} + \frac{\partial V_q}{\partial q}^t = 0$ where $C_q(q, \dot{q})\dot{q} = \frac{\partial}{\partial q}\left(M_q(q)\dot{q}\right)\dot{q} - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^t M_q(q)\dot{q}\right)$. The term $C_q(q, \dot{q})\dot{q}$, which is quadratic in $\dot{q}$, accounts for centrifugal and Coriolis forces. Let the r-dimensional vector $D_q(q, \dot{q})$ represents the external forces, then the controlled equations of motion in local coordinates are of the form $u_q = M_q(q)\ddot{q} + N_q(q, \dot{q})$, where

$$N_q(q, \dot{q}) = C_q(q, \dot{q})\dot{q} + \frac{\partial V_q}{\partial q}^t(q) - D_q(q, \dot{q}) \tag{2}$$

is a smooth r-dimensional vector and $u_q(\cdot)$ is the control. The system is conservative when the control and the extrenal forces are assumed to be zero. To summarize, we have that a controlled mechanical system is a system with equations of motion given in local coordinates by

$$u_q(t) = M_q(q(t))\ddot{q}(t) + N_q(q(t), \dot{q}(t)) \tag{3}$$

where $M_q(q)$ and $N_q(q, \dot{q})$ are defined as above. All objects are assumed to be smooth. Let us first rewrite the equations of motion using the velocities in the body-fixed frame instead of in the inertial frame. The motivation to take this point of view comes from the application we consider in this paper. Indeed, we assume that the acceleration of the underwater vehicle is provided through the use of thrusters that are fixed to the vehicle. The inputs are then expressed in the body-fixed frame. We introduce $v = P^{-1}(q)\dot{q}$ where $P(q)$ is a smooth $r \times r$ invertible matrix. The following relations hold:

$$\dot{q} = P(q)v, \quad \ddot{q} = P(q)\dot{v} + \dot{P}(q)v. \tag{4}$$

If we define: $M_q(q) = P^{-t}(q)M(q)P^{-1}(q)$, $C_q(q, \dot{q}) = P^{-t}(C(q, v) - M(q)P^{-1}(q)\dot{P}(q))P^{-1}(q)$, $D_q(q, \dot{q}) = P^{-t}(q)D(q, v)$, $\frac{\partial V_q}{\partial q}(q) = P^{-t}\frac{\partial V}{\partial q}(q)$, $u_q = P^{-t}(q)u$, then the equations of motion in this new system of coordinates are given by

$$M(q)\dot{v} + C(q, v)v + D(q, v) + \frac{\partial V}{\partial q}(q) = u. \tag{5}$$

Based on the previous definitions, we derive in this section the equations of motion for marine vehicles in 6 degrees of freedom in the body-fixed frame. We denote by $\eta = (x, y, z, \phi, \theta, \psi)^t$ the position and orientation of the vehicle with respect to the earth-fixed reference frame, the coordinates ϕ, θ, ψ being the Euler angles for the body frame. The coordinates corresponding to translational and rotational velocities in the body frame are $\nu = (u, v, w, p, q, r)^t$ (notice that u represents here the surge and not the control as it was the case in the previous section). We define $\eta_1 = (x, y, z)^t, \eta_2 = (\phi, \theta, \psi)^t$ and $\nu_1 = (u, v, w)^t, \nu_2 = (p, q, r)^t$. If J represents the linear and angular velocity transformations, we have that

$$\dot{\eta} = J(\eta)\nu \tag{6}$$

Let $L = T - V$ be the Lagrangian, with $T = T_{RB} + T_A$; respectively the rigid-body kinetic energy and the fluid kinetic energy. The potential V is defined implicitly by

$$\frac{\partial V}{\partial \eta} = J(\eta)g(\eta) \tag{7}$$

where $g(\eta)$ represents the restoring forces (gravitational forces and moments). Using the quasi-Lagrange equations (a generalization of Kirchoff's equations when considering a non zero potential), we obtain

$$\dot{\nu} = -M^{-1}(C + D)\nu - M^{-1}g + M^{-1}\tau \tag{8}$$

$$\dot{\eta} = J(\eta)\nu \tag{9}$$

where M is the inertia matrix, C the coriolis and centripetal matrix, D the damping forces and the variable τ represents the control. See Fossen (1994); Chyba and Haberkorn (2005) for more details.

3. SINGULAR EXTREMALS

In this section we will derive necessary conditions for a trajectory to be a singular extremals in the time minimum problem. Our computations are based on the maximum principle, see Bonnard and Chyba (2003).

From now on, we assume that the configuration manifold $\mathcal{Q}$ is the Euclidean space $\mathbb{R}^6$. The equations of motion for an underwater vehicle, see (8) and (9) can be rewriten as a control system

$$\dot{x} = f(x) + B(x)\tau \tag{10}$$

as follows. Introduce $x = (x_1, x_2)$ where $x_1 = \eta$ and $x_2 = \nu$. Notice that the matrix P is for the underwater vehicle application given by the matrix $J(\eta)$ (assuming we are not at a configuration for which this matrix is singular) and that the control is acting in the directions of the body- frame axis. We have that the drift f is

$$f(x) = \begin{pmatrix} J(\eta)\nu \\ -M^{-1}(C+D)\nu - M^{-1}g \end{pmatrix} \tag{11}$$

while the control vector fields are given by

$$B(x) = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} \tag{12}$$

Since the controls represent the thrust, we assume bounds as follow:

$$\mathcal{U} = \{\tau \in I\!R^6;\ |\tau_i| \le \tau_i^{\max},\ i = 1, \cdots, 6\}. \tag{13}$$

Let $x_0, x_T \in I\!R^{12}$ be the initial and final states. We assume there is an admissible time optimal control $\tau : [0,T] \to \mathcal{U}$ such that the corresponding time optimal trajectory $x(\cdot)$, a solution of (10), is defined on $[0,T]$ and steers the system from x_0 to x_T. The maximum principle states that there exists an absolutely continuous vector $\lambda : [0,T] \to I\!R^{12}$, $\lambda(t) \neq 0$ for all t, such that the following conditions hold almost everywhere:

$$\dot{x}_j(t) = \frac{\partial H}{\partial \lambda_j}(\lambda(t), x(t), \tau(t)), \tag{14}$$

$$\dot{\lambda}_j(t) = -\frac{\partial H}{\partial x_j}(\lambda(t), x(t), \tau(t)) \tag{15}$$

for $j = 1, \cdots, 12$, where $H(x, \lambda, \tau) = \lambda_\eta^t J(\eta)\nu + \lambda_\nu^t(-M^{-1}(C+D)\nu - M^{-1}g) + \lambda_\nu^t M^{-1}\tau$ is the Hamiltonian function with $\lambda = (\lambda_\eta, \lambda_\nu) \in I\!R^6 \times I\!R^6$, and the maximum condition holds:

$$H(\lambda(t), x(t), \tau(t)) = \max_{v \in \mathcal{U}} H(\lambda(t), x(t), v). \tag{16}$$

Moreover, the maximum of the Hamiltonian is constant along the solutions of (14) and must satisfy $H(\lambda(t), x(t), \tau(t)) = \lambda_0$, $\lambda_0 \ge 0$. A triple (x, λ, τ) which satisfies the maximum principle, in the sense just stated, is called an extremal. Its projection on the state space $x(\cdot)$ is said to be a geodesic, and the vector function $\lambda(\cdot)$ an adjoint vector. When the constant λ_0 is zero, the extremal is said to be abnormal.

Let us denote the column vector of the inverse matrix M^{-1} by $M_1^{-1}, \cdots, M_6^{-1}$, $M^{-1} = (M_1^{-1}, \cdots, M_6^{-1})$. The switching functions are defined by $\phi_i(t) = \lambda_\nu^t M_i^{-1}$, $i = 1, \cdots, 6$. Due to the maximisation condition and the bounds on the controls, the switching functions play a major role in the structure of the time optimal trajectories. Since the M_i^{-1} are constant, the switching functions are linear combinations of the components of the adjoint vector corresponding to the velocity variables. From the maximum condition, the following holds almost everywhere for $i = 1, \cdots, 6$:

$$\tau_i(t) = -\tau_i^{\max} \text{ if } \varphi_i(t) < 0 \tag{17}$$

$$\tau_i(t) = \tau_i^{\max} \text{ if } \varphi_i(t) > 0, \tag{18}$$

If there exists a nontrivial interval $[t_1, t_2] \subset [0,T]$ such that $\varphi_i(t)$ is identically zero, the corresponding extremal is called τ_i-singular on $[t_1, t_2]$. The τ_i component of the control is then called singular on $[t_1, t_2]$. An extremal is called totally singular on $[t_1, t_2]$ if it is τ_i-singular on $[t_1, t_2]$ for each i. The maximum principle implies that if $\varphi_i(t) \neq 0$ for almost all $t \in [0,T]$ the u_i component of the control is bang-bang, which means that it takes its values in $\{\alpha_i, \beta_i\}$ for almost every $t \in [0,T]$. Assume that the u_i component of the control is bang-bang; then $t_s \in [0,T]$ is called a switching time for u_i if, for each interval of the form $]t_s - \varepsilon, t_s + \varepsilon[\cap [t_1, t_2]$, $\varepsilon > 0$, there is no constant c such that $\tau_i(t) = c$ for almost all $t \in [t_1, t_2]$. An extremal is called τ_i-regular if τ_i is bang-bang with at most a finite number of switchings. An extremal is said to be regular if it is τ_i-regular for all i.

A necessary condition for an extremal to be τ_i-singular on a given interval is that the corresponding switching function φ_i is identically zero along this interval. Let us compute the derivatives of φ_i to derive additional conditions. Denote $\hat{M}_i^{-1} = \begin{pmatrix} 0 \\ M_i^{-1} \end{pmatrix}$ and using the fact that $[M_i^{-1}, M_j^{-1}] = 0$ for each pair i, j, we have that $\dot{\phi}_i$ is an absolutely continuous function defined by:

$$\dot{\varphi}_i(t) = -\lambda^t(t)[f, \hat{M}_i^{-1}](x(t)). \tag{19}$$

where

$$[f, \hat{M}_i^{-1}] = \begin{pmatrix} J(\eta)M_i^{-1} \\ \dfrac{\partial}{\partial \nu}(-M^{-1}(C+D)\nu)M_i^{-1} \end{pmatrix} \tag{20}$$

The second derivatives is a measurable bounded function such that almost everywhere the following holds:

$$\ddot{\varphi}_i(t) = \lambda^t ad_f^2 \hat{M}_i^{-1} + \sum_{j=1}^{n} \lambda^t(t)[\hat{M}_j^{-1}, [f, \hat{M}_i^{-1}]](x(t))\tau_j(t) \tag{21}$$

where $ad_f^2 \hat{M}_i^{-1}$ stands for $[f, [f, \hat{M}_i^{-1}]]$. The first n components of the vector field $[f, \hat{M}_i^{-1}]$ depend exclusively on η, then the first n components of $[\hat{M}_j^{-1}, [f, \hat{M}_i^{-1}]]$ are zeroes. An easy compuation shows that

$$[\hat{M}_j^{-1}, [f, \hat{M}_i^{-1}]] = \begin{pmatrix} 0 \\ S_{ji}(\eta, \nu) \end{pmatrix} \tag{22}$$

where $S_{ji} = \frac{\partial}{\partial \nu}(\frac{\partial}{\partial \nu}(-M^{-1}(C + D)\nu)M_i^{-1})M_j^{-1}$.

In the sequel we will focus on the role of the Lie bracket $[\hat{M}_i^{-1}, [f, \hat{M}_i^{-1}]]$. This vector field is indeed related to the order of the singular extremal and under some assumption on the system to the existence of a decoupling vector field. We define the order of τ_i-singular extremal as follows. Let q be such that $\frac{d^{2q}}{dt^{2q}}\varphi_i$ is the lowest order derivative in which τ_i appears explicitely with a nonzero coefficient. The number q is the order of the singular control τ_i. The above definition lies on the fact that it is a well known result that a singular control τ_i first appears explicitely in an even order derivative of φ_i. From (21), we have that a singular extremal is of order at least 2 if $[\hat{M}_i^{-1}, [f, \hat{M}_i^{-1}]] = 0$. This property is related with the existence of chattering arcs, see Chyba and Haberkorn (to appear). In Chyba and Haberkorn (2005), we analyze the vector fields involved in (21) under the following assumptions. No damping and the vehicle is neutrally buoyant with coincident center of gravity and center of buoyancy. We obtained that in this case the vector fields $[\hat{M}_i^{-1}, [f, \hat{M}_i^{-1}]]$ are identically zero. Let us generalise these computations to the following case. We assume first that the center of gravity of the vehicle is located at the origin. Following Chyba and Haberkorn (submitted), we have that

$$M = \begin{bmatrix} 3/2\,m & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/2\,m & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/2\,m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & -I_{xy} & -I_{xz} \\ 0 & 0 & 0 & -I_{xy} & I_y & -I_{yz} \\ 0 & 0 & 0 & -I_{xz} & -I_{yz} & I_z \end{bmatrix} \tag{23}$$

where m is the mass of the AUV, and $I_{.}$ and $I_{..}$ are inertia factors. The coriolis matrix C is of the form

$$\left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & mw - Z_{wd}\,w & -mv + Y_{vd}\,v \\ -mw + Z_{wd}\,w & -0 & mu - X_{ud}\,u \\ mv - Y_{vd}\,v & -um + X_{ud}\,u & -0 \end{matrix} \right. \tag{24}$$

$$\left. \begin{matrix} 0 & mw - Z_{wd}\,w & -mv + Y_{vd}\,v \\ Z_{wd}\,w - mw & -m\,0 & mu - X_{ud}\,u \\ mv - Y_{vd}\,v & -mu + X_{ud}\,u & 0 \\ 0 & C_{4,5} & C_{4,6} \\ C_{5,4} & 0 & C_{5,6} \\ C_{6,4} & C_{6,5} & 0 \end{matrix} \right] \tag{25}$$

where the coefficients $C_{.,.}$ are:

$$\begin{cases} C_{4,5} &= I_z r - I_{yz} q - I_{xz} p - N_{rd} r, \\ C_{4,6} &= I_{yz} r + I_{xy} p - I_y q + M_{qd} q \\ C_{5,4} &= I_{yz} q + I_{xz} p - I_z r + N_{rd} r, \\ C_{5,6} &= I_{xz} r - I_{xy} q + I_x p - K_{pd} p \\ C_{6,4} &= I_y q - I_{yz} r - I_{xy} p - M_{qd} q, \\ C_{6,5} &= I_{xy} q - I_x p - I_{xz} r + K_{pd} p \end{cases} \tag{26}$$

The damping forces D are assumed of the form $-diag(X_u + X_{uu}|u|, Y_v + Y_{vv}|v|, Z_w + Z_{ww}|w|, K_p + K_{pp}|p|, M_q + M_{qq}|q|, N_r + N_{rr}|r|)$ where (X_u, X_{uu}), (Y_v, Y_{vv}), and (Z_w, Z_{ww}) are the drag coefficients for pure surge, sway, and heave, resp. And, (K_p, K_{pp}), (M_q, M_{qq}), (N_r, N_{rr}) are the drag coefficients for pure roll, pitch, and yaw, resp.

Finally, the restoring forces are represented by the vector $g(\eta)$:

$$g = \begin{bmatrix} (W_g - B)\sin\theta \\ \\ (-W_g + B)\cos\theta\sin\phi \\ (-W_g + B)\cos\theta\cos\phi \\ (y_B B)\cos\theta\cos\phi + (z_B B)\cos\theta\sin\phi \\ (-z_B B)\sin\theta + (-x_B\, B)\cos\theta\cos\phi \\ (x_B\, B)\cos\theta\sin\phi - (-y_B\, B)\sin\theta \end{bmatrix} \tag{27}$$

where (x_B, y_B, z_B) stands for the center of buoyancy, W_g is the mass of the vehicle, and B is the buoyancy force on the vehicle.

To draw a correspondance between singular extremals and the existence of decoupling vector fields, let us first look at the situation when the restoring forces and the drag are taken as zero. It follows that

$$S_{ii}(\eta, \nu) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s_i^4 \\ s_i^5 \\ s_i^6 \end{pmatrix} \tag{28}$$

where s_i^4, s_i^5, s_i^6 are functions that depend on the inertia coefficients. We have that $s_i^4 = s_i^5 = s_i^6 = 0$ for $i = 1, 2, 3$ and, if we assume I_{xy}, I_{xz}, I_{yz} to be zero then the vector S_{ii} is identically zero for any i. The conclusion is that the order of the τ_i-singular extremal is at least 2 for $i = 1, 2, 3$ as well as for $i = 4, 5, 6$ under the assumption that $I_{xy} = I_{xz} = I_{yz} = 0$. Since under our assumptions we have no potential or extrenal forces, we can applied the results for decoupling vector fields described in Bullo and Lewis (2004). Consider a mechanical system with no potential or external forces written under the form of a control system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i, \tag{29}$$

where $x = (x_1, \dot{x}_1)$ and $f_i = \begin{pmatrix} 0 \\ Y_i \end{pmatrix}$, see Lewis and Murray (1997) for instance for more information on the form of f_0 and the f_i. A vector field V is a decoupling vector field for (29) if and only if the distribution generated by V and $\langle V : V \rangle$, where $\langle . : . \rangle$ represents the symmetric product, is a subdistribution of the input distribution generated by the vector fields Y_i. It means that for any absolutely continuous control w_i and correspoding solution x_1 of the system $\dot{x}_1 = V(x_1(t))w_i(t)$ there exists a measurable bounded control u_i such that x_1 is a controlled trajectory for (29).

Clearly, for a fully actuated situation, any vector field is a decoupling vector field. However, in our situation we can draw stronger conclusions. First notice that since $vlift(\langle Y_i : Y_j \rangle) = [f_i, [f_0, f_i]]$ where $f_i = \begin{pmatrix} 0 \\ Y_i \end{pmatrix}$, we have that if $[f_i, [f_0, f_i]] = 0$ then Y_i is decoupling. What is interesting is that Y_i is decoupling for the reduced system corresponding to (29) with $u_j = 0$ for $j \neq i$: $\dot{x} = f_0(x) + f_i(x)u_i$.

Let us apply these results to the underwater vehicle situation. We have to be a little carefull about the choice of coordinates we have made. Indeed, for the underwater vehicle application, we expressed the velocity in the body frame coordinates which is not the case in (29). However, the condition $[\hat{M}_i^{-1}, [f, \hat{M}_i^{-1}]] = 0$ is related to the order of the singular extremal and then will still be true in the system of coordinates $(\eta, \dot{\eta})$. The inputs of the system in coordinates (η, ν) are simply obtained by the transformation $J(\eta)$ from the inputs of the system expressed in coordinates $(\eta, \dot{\eta})$.

We have the following for the underwater vehicle. If the singular extremal is at least of order 2, then the input vector fields $f_i = \begin{pmatrix} 0 \\ Y_i \end{pmatrix}$ expressed in the inertial coordinate system are decoupling vector fields for the system

$$\dot{x} = f(x) + f_i(\eta)\tau_i. \tag{30}$$

This means that any trajectory solution of $\dot{\eta}(t) = f_i(\eta)w_i(t)$ associated to an absolutely continuous control is a solution of (30) for a measurable bounded control. In particular, we can decouple the motion of the underwater vehicle in each of the 6 degrees of freedom of the system. From a practical point of view, we have obtained that a motion along any axis in the inertial frame can be realized using only a single input when the system is expressed in the coordinates $(q, \dot{q})$ or can be realized using a combination of the three inputs $M_1^{-1}, M_2^{-1}, M_3^{-1}$ when the velocities are expressed in the body frame coordinates. These motions are called basic translations in the inertial coordinate frame.

In the final version we will discuss the implications in the trajectory design problem and we will discuss the impact of the additional terms neglected such as the drag on the order of a singular extremals and then on a possible extension of the notion of decoupling vector fields to systems with potential forces.

4. CONCLUSION

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