Time optimal Paths for a mobile robot with one trailer

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Abstract

Optimal paths are in many ways interesting to motion planning. Not only they are obviously the most interesting ones with respect to the optimized criterion, but also they offer a way of studying the topological aspects associated to the controllability of the system. Many works have been carried out successfully on the optimal paths for unicyle and carlike system. This paper aims to extend those results when adding trailers. Characterizing the time optimal paths for a mobile robot with n-trailers is an ambitious task. As a first step, in this paper we give an explicit description of the abnormal and singular extremals for a mobile robot with one trailer. This will be done with the help of the Maximum Principle.

1 Introduction

This work has been motivated by motion planning problems for wheeled mobile robots. For these robots, the dimension of the set of admissible velocities at each time is reduced by some kinematic constraints. These kinematic constraints stem from the mechanical constraints of non slippage of the wheels which for example, make impossible lateral translations. Since this restriction in the tangent space cannot be expressed as a reduction of the state space to a submanifold, it defines a *nonholonomic* constraint. The motion planning problem for nonholonomic systems is known to be particularly challenging.

Indeed, nonholonomy implies that any curve including a reachable configuration q does not correspond to a valid trajectory to reach q (since it can violate the kinematic constraints). Nevertheless, usually there exists an infinite number of admissible trajectories to reach q. Therefore, a first step for solving the motion planning problem for a given nonholonomic system is to find a *steering method* for that system regardless of obstacles. That means, an *automatic* way of picking up a *unique* trajectory between any two extremal configurations among the set of the *admissible* trajectories. The optimality of this trajectory with respect to any given criteria is obviously an interesting property. Think of length optimal paths that can reduce the risk of hitting the surrounding obstacles, time optimal paths, etc.

Of course, optimal control is not the only way of defining a steering method. For some specific systems there are some specific ways of defining an open-loop control leading to a steering method (see [13] for chained form systems, [5, 10] for flat systems, etc). However for many optimality criteria (such as time for example , optimal trajectories have a very interesting property : even if we restrict the authorized motions of the system to the set of optimal motions, we still conserve important properties of the system such as the *small-time local controllability.* Hence, a steering method based on optimality verifies what is called the *topological property* TP [9]. Now, let us consider the general nonholonomic motion planning problem that consists in finding a path which is kinematically feasible and moreover avoid obstacles. Almost all existing planners attempt to solve the problem by integrating a steering method into a global geometric obstacle avoidance scheme. Then, the topological property of the steering method is the key point that guarantees the convergence of the whole algorithm [9]. While designing a steering method is a hard task, designing one that respects TP is even harder. Finding the set of the optimal trajectories for a system may be hard but one can prove in advance that the corresponding steering method will respect TP (see [9] for example . Therefore it can allow to steer the system even in presence of obstacles.

Based on Pontryagin's maximum principle (PMP) several works have been done on optimal paths for mobile robots; see [11, 12, 1] for instance. These works concern the case of the unicycle. In this paper we take an interest in the more general family of tractor-trailers systems. Any result on these systems has not only a practical repercussion for tractor-trailers systems but also a more theoretical one. Indeed, equivalences exist between the tractor-trailers systems and any driftless flat system with two inputs and also any chained form system under Goursat form with two inputs (see [8]).

This paper is concentrated on the study of a tractor with one trailer. Regarding optimality, the PMP implies some necessary conditions on the optimal paths. Nevertheless, the informations obtained directly from the PMP are not enough to obtain a sufficient family of optimal paths and an optimal *synthesis*. More informations have to be supplemented by geometric techniques. The use of modern geometric optimal control in such problems is remarkably illustrated in [12] where the authors study the shortest paths for the Reed-Shepp car. From the PMP we usually just get a local characterization of extremals. An optimal path between two configurations corresponds in general to a concatenation of several extremals. The PMP gives no direct information on the way to connect the elementary pieces of extremals or a bound on the number of switches. In this paper we classify all extremals (including abnormal and singular ones) for a tractor with one trailer and characterize each of them. We also present the switching structure equations (SSE) which not only allows us to realize our classification but also are the key equations to understand the optimal strategy for concatenating extremals. These are the necessary first steps in order to find a sufficient family of optimal paths. This will be hopefully done in a forthcoming article.

We first recall briefly in section 2 Pontryagin's result, then the third section is devoted to the study based on the PMP of the time optimal trajectories for a mobile robot with a trailer.

2 Maximum Principle

We consider the time optimal problem for systems of the following form :

$$\dot{q}(t) = X(q(t)) + Y_1(q(t))u_1(t) + ... + Y_m(q(t))u_m(t) \qquad (1)$$

with $q \in \mathbf{R}^n$, $X(q), Y_i(q)$ smooth vector fields, $u(t) \in U = \{(u_1, ...u_m) \in \mathbf{R}^m, u_i \in [-a_i, a_i]\}$. We assume that u is measurable and bounded.

A classical tool coming from the optimal control theory and giving information about the optimal strategy is the Pontryagin maximum principle (PMP). To state this principle let us define the *Hamiltonian* function $H : \mathbf{R}^n \times \mathbf{R}^n_* \times U \to \mathbf{R}$ by

$$H(q,p,u) = < p, X(q) > + \sum_{i=1}^{m} < p, Y_i(q) > u_i \qquad (2)$$

where $<,>$ is the usual inner product in $\mathbf{R}^n$. According to Pontryagin's Maximum Principle [7], if a control $u(t)$ is time optimal on $[0,T]$, $q(t)$ being the corresponding trajectory, then there exists an *adjoint vector* $p(t) \in \mathbf{R}^n_*, 0 \leq t \leq T$, absolutely continuous, such that for almost all t :

$$\frac{dq}{dt}(t) = \frac{\partial H}{\partial p}(q(t),p(t),u(t)) \qquad (3)$$

$$\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}(q(t),p(t),u(t)) \qquad (4)$$

$$H(q(t),p(t),u(t)) = \max_{v \in U} H(q(t),p(t),v) = -H_0 \qquad (5)$$

where H_0 is a constant ≥ 0.
The projection q of a triple (q,p,u) solution of (3),(4) and (5) is called *an extremal*, and u the extremal control. Then, the necessary conditions of optimality can be formulate in the following way : all optimal trajectories must be lift into an extremal bitrajectory (q,p). If (q,p,u) is a solution of the PMP defined on $[0,T]$, the function $\phi_i : t \to \phi_i(t) = < p(t), Y_i(q(t)) >$ will be called the *i-th switching function*, $i = 1, ..., m$. From (5), it is clear that if $\phi_i(t) \neq 0$, then $u_i(t) = sign(\phi_i(t))a_i$. The control $u_i(t)$ is called *bang-bang* if it is a piecewise constant function on $[0,T]$ taking its values in $\{-a_i, +a_i\}$. An extremal is called *regular* if for all i, $\phi_i \neq 0$ except on a set of isolated values of t. In this case, $u_i(t) = sign(\phi_i(t))a_i$ except for a finite number of t. An extremal is *singular* if there exists i such that $\phi_i(t) \equiv 0$ on a non empty interval $[\tau_1, \tau_2] \subset [0,T]$. Characterizing singular extremals requires more work. These extremals play an important role in the design of the optimal trajectories. In Section 4 we present a characterization of regular and singular extremals for a tractor with one trailer.

3 Mobile robot with one trailer

3.1 The model

A configuration q of our system may be given by 4 parameters (x, y, θ, β) represented on Figure 1. Expressing the non slippage of the wheels in these coordinates leads to the following differential equations for the system :

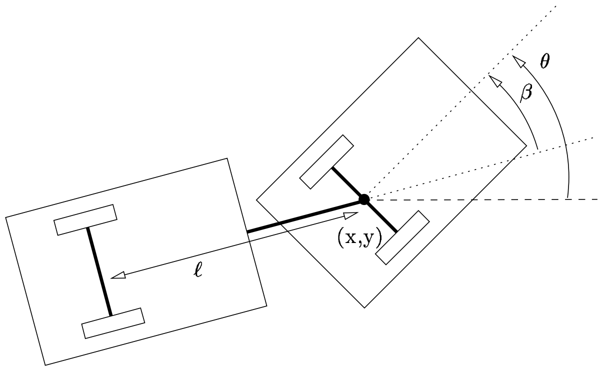

Figure 1: The tractor-trailer system coordinates

$$\dot{q} = X_1(q)u_1 + X_2(q)u_2 \tag{6}$$

with :

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\frac{\sin\beta}{\ell} \end{pmatrix} \quad \text{and} \quad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{7}$$

The control of the system are $|u_1| \leq 1$ the tangential velocity of the tractor and $|u_2| \leq 1$ its angular velocity. For sake of simplicity we set for the sequel $l = 1$. The vectors X_1 and X_2 are the vector fields of the system.

Let us recall here the notion of *Lie brackets* of vector fields which will be of use for the rest of the paper. Given $X, Y : \Omega \to \mathbf{R}^n$ two vector fields of class C^1 defined on an open set $\Omega \subset \mathbf{R}^n$, the *Lie brackets* $[X, Y]$ is defined as the vector field given by :

$$[X, Y](q) = \frac{\partial Y}{\partial q} X(q) - \frac{\partial X}{\partial q} Y(q)$$

where $q \in \Omega$. The Lie algebra of vector fields generated by a family of m-smooth vector fields $F = \{X_1, ...X_m\}$ is defined as the smallest linear space $Lie(F)$ of smooth vector fields such that if $X, Y \in Lie(F)$, then $[X, Y] \in Lie(F)$.

Controllability Before trying to find an optimal path to reach a configuration one has to make sure that this configuration is reachable. If we consider $F = \{X_1, X_2\}$ for our system, it is easy to check that at any configuration $dimLie(F) = 4$ which is the dimension of the configuration space. One says that our system respect the Lie Algebra Rank Condition. Furthermore, our system is symmetric and the configuration is a connected space. These three properties imply that our system is controllable, which means any configuration is reachable. For more detail refer to [6] where the controllability for a mobile robot with an arbitrary number of trailers has been established.

Existence of optimal trajectories To find results about the existence of optimal trajectories in general, the reader can refer for example to [3]. In our situation we can notice that as the vector fields X_1, X_2 are bounded we have the completeness condition for our trajectories. Moreover the control set is compact and convex and given any two points q_1, q_2 there is a trajectory from q_1 to q_2. These conditions imply in a classical way the following property :

Proposition 1 *For the time optimal problem associated to system (6), given any two points $q_1, q_2 \in \mathbf{R}^4$, there exists a time-optimal trajectory from q_1 to q_2.*

4 Time-optimal extremals and SSE

4.1 The Switching Structure Equations

Let us denote $p = (p_x, p_y, p_\theta, p_\beta)$, the Hamiltonian function corresponding to the mobile robot with a trailer is given from (2) by the following formula :

$$H(p, q, u) = (p_x \cos\theta + p_y \sin\theta - p_\beta \sin\beta)u_1 + (p_\theta + p_\beta)u_2 \tag{8}$$

We denote by ϕ_1, ϕ_2 the switching functions

$$\phi_1 = p_x \cos\theta + p_y \sin\theta - p_\beta \sin\beta, \phi_2 = p_\theta + p_\beta$$

Recall that along an optimal path, the Hamiltonian has a maximal constant value :

$$\phi_1 u_1 + \phi_2 u_2 = -H_0 \tag{9}$$

We can deduce immediately the following properties for $i = 1, 2$:

P1: If $\phi_i(t) > 0$ then $u_i(t) = +1$
P2: If $\phi_i(t) < 0$ then $u_i(t) = -1$

Hence, the sign of $\phi_i(t)$ has a great importance in the determination of the optimal strategy. In order to study the sign of the switching functions, let us take an interest in their derivative. We have seen that for $i = 1, 2$, $\phi_1(t) =< p(t), X_i(q(t)) >$. Let us introduce the following functions:

$$\begin{array}{rcl} \phi_3 & = & \langle p, [X_2, X_1] \rangle \\ \phi_4 & = & \langle p, [X_1, [X_2, X_1]] \rangle \\ \phi_5 & = & \langle p, [X_1, [X_1, [X_2, X_1]]] \rangle, \\ \phi_6 & = & \langle p, [X_2, [X_1, [X_1, [X_2, X_1]]]] \rangle \end{array} \tag{10}$$

Then we get the following proposition on SSE :

Proposition 2 *The Switching Structure Equations are given by*

$$\begin{array}{ll} \dot{\phi}_1 = \phi_3 u_2 & \dot{\phi}_2 = -\phi_3 u_1 \\ \dot{\phi}_3 = -\phi_1 u_2 + \phi_4 u_1 & \dot{\phi}_4 = \phi_5 u_1 \\ \dot{\phi}_5 = \phi_4 u_1 + \phi_6 u_2 & \dot{\phi}_6 = -\phi_5 u_2 \end{array} \quad (11)$$

where ϕ_i for $i = 3..6$ are given by (10),

$$\text{for } i = 1,2 \ \text{ if } \phi_i(t) \neq 0 \text{ then } u_i(t) = \text{sign}\phi_i(t) \quad (12)$$

$$\mid \phi_1(t) \mid + \mid \phi_2(t) \mid + H_0 = 0 \quad (13)$$

where H_0 is given by the formula 5,

$$\mid \phi_1(t) \mid + \mid \phi_2(t) \mid + \mid \phi_3(t) \mid + \mid \phi_4(t) \mid \neq 0 \quad (14)$$

and

$$\phi_5^2 + \phi_6^2 = \phi_4^2 \quad (15)$$

proof: The SSE (11) is computed using the following fact. If Z is a smooth vector field in $R^2 \times S^1 \times S^1$ and (q,p) an extremal lift corresponding to the controls (u_1, u_2), then the derivative with respect to t of the function $t \to \langle p(t), Z(q(t)) \rangle$ is given by $\langle p(t), [X_1, Z](q(t)) \rangle u_1(t) + \langle p(t), [X_2, Z](q(t)) \rangle u_2(t)$. The equations (12) and (13) are direct consequences of PMP. For the equation (14), one has to notice that $X_1, X_2, [X_1, X_2], [X_1, [X_2, X_1]]$ is a basis of the tangent space and $\phi_1, \cdots, \phi_4$ are coordinates of $p \in \mathbf{R}^4_*$ in this basis. Finally, the equation (15) can be obtained by an explicit computation of ϕ_i's for our system using (10).

4.2 Regular extremals

Following the PMP, the regular extremals are fully determined by the switching functions. The regular extremals are concatenations of one of the four cases described below:

Case 1: $u_1 \equiv 1$, $u_2 \equiv 1$
Case 2: $u_1 \equiv -1$, $u_2 \equiv 1$
Case 3: $u_1 \equiv 1$, $u_2 \equiv -1$
Case 4: $u_1 \equiv -1$, $u_2 \equiv -1$
Assume the starting point at $(x_0, y_0, \theta_0, \beta_0)$. We can integrate the equations of the motion of our system given by 6 and we obtain the following results :

Case 1

$$\begin{array}{c} x(t) = x_0 + \sin(t+\theta_0) - \sin(\theta_0) \\ y(t) = y_0 - \cos(t+\theta_0) + \cos(\theta_0) \\ \theta(t) = t + \theta_0 \qquad \beta(t) = 2\arctan(\frac{t-2+C}{t+C}) \end{array}$$

where $C = \frac{2}{1-\tan\frac{\beta(0)}{2}}$.

Case 2

$$\begin{array}{c} x(t) = x_0 - \sin(t+\theta_0) + \sin(\theta_0) \\ y(t) = y_0 + \cos(t+\theta_0) - \cos(\theta_0) \\ \theta(t) = t + \theta_0 \qquad \beta(t) = -2\arctan(\frac{t+2+C}{t+C}) \end{array}$$

where $C = \frac{2}{\tan(-\frac{\beta(0)}{2})-1}$.

Case 3

$$\begin{array}{c} x(t) = x_0 - \sin(-t+\theta_0) + \sin(\theta_0) \\ y(t) = y_0 + \cos(-t+\theta_0) - \cos(\theta_0) \\ \theta(t) = -t + \theta_0 \qquad \beta(t) = -2\arctan(\frac{t-2+C}{t+C}) \end{array}$$

where $C = \frac{2}{1-\tan(-\frac{\beta(0)}{2})}$.

Case 4

$$\begin{array}{c} x(t) = x_0 + \sin(-t+\theta_0) - \sin(\theta_0) \\ y(t) = y_0 - \cos(-t+\theta_0) + \cos(\theta_0) \\ \theta(t) = -t + \theta_0 \qquad \beta(t) = 2\arctan(\frac{t+2+C}{t+C}) \end{array}$$

where $C = \frac{2}{\tan\frac{\beta(0)}{2}-1}$.

From our study we have immediately the following lemma :

Lemma 1 *Along a time optimal regular trajectory, the motion of x and y is an arc of circle.*

4.2.1 Abnormal extremals

Definition 1 *Extremals for which $H_0 = 0$ are called abnormal extremals.*

We call *trivial extremal*, the path corresponding to $u_1 = u_2 \equiv 0$ which is also obviously an optimal.

Lemma 2 *If a non trivial extremal is abnormal, then $u_1 \equiv 0$ and $\phi_4 = cte$.*

proof: From equation (13), along an abnormal extremal we have $\phi_1 = \phi_2 \equiv 0$. Then $\dot{\phi}_1 \equiv 0$ and if the extremal is not trivial the equation (11) implies $\phi_3 \equiv 0$, hence $\dot{\phi}_3 \equiv 0$ and then $\phi_4 u_1 \equiv 0$. From (14) we deduce that $\phi_4 \neq 0$ therefore $u_1 \equiv 0$ along a non trivial abnormal extremal and so (11) implies $\dot{\phi}_4 \equiv 0$.

Proposition 3 *The only abnormal non trivial optimal trajectories are of one of the two following types :*
A^+ : $u_1 \equiv 0$, $u_2 \equiv 1$
A^- : $u_1 \equiv 0$, $u_2 \equiv -1$

proof : From Lemma 2, on such an optimal $u_1 \equiv 0$. The equations (1) of the system are then reduced to :

$$x(t) = x_0, \ \ y(t) = y_0, \ \ \dot{\theta}(t) = \dot{\beta}(t) = u_2$$

if θ_{end} and β_{end} are the final values of the two last coordinates, it is easy to prove that the fastest way of reaching them is to apply at each time the maximal absolute value of the angular velocity u_2.

Lemma 3 *Non trivial extremal can have common zero for the switching functions only if it is an abnormal extremal.*

proof: If $\phi_1(t) = \phi_2(t) = 0$, then from (13) we have $H_0 = 0$.

4.2.2 Singular extremals

Definition 2 *An extremal defined on $[0, T]$ is called singular if at least one of the switching function is zero on a non empty interval $[\tau_1, \tau_2] \subset [0, T]$.*

- $\phi_1 \equiv 0$ on $[\tau_1, \tau_2]$: As $\phi_1 \equiv 0$, we have $\dot{\phi_1} \equiv 0$ and (11) implies that we have two situations to consider: $\phi_3 \equiv 0$ or $u_2 \equiv 0$. If $u_2 \equiv 0$, we have from (9) that the extremal is abnormal. From Lemma 2 u_1 is also null. This is a trivial extremal. Therefore, we have only to consider cases where $u_2 \neq 0$. Then $\phi_3 \equiv 0$ and from (11) we obtain $\dot{\phi_2} \equiv 0$. If ϕ_2 is also null, the extremal is abnormal again. Therefore, we just have to consider cases where ϕ_2 is a non null constant. Then, (12) implies $u_2 \equiv \pm 1$. Moreover we have $\dot{\phi_3} = \phi_4 u_1 \equiv 0$. If we assume $u_1 \neq 0$, then by equation (11) we can prove that $\phi_i \equiv 0$ for $i = 4, 5, 6$. One can prove that in this case $p_x = p_y = p_\beta \equiv 0$ and p_θ is constant. Conversely, one can check that $(0, 0, P_\theta, 0)$ with P_θ a non null constant is a adjoint vector for any trajectory. Furthermore, the corresponding extremal is ϕ_1-singular ($\phi_1 \equiv 0$) and $\phi_2 \equiv P_\theta$. Which implies that any trajectory with $u_2 \equiv \pm 1$ and u_1 chosen arbitrarily is a ϕ_1-singular extremal.

- $\phi_2 \equiv 0$ on $[\tau_1, \tau_2]$: If $\phi_2 \equiv 0$, then $\dot{\phi_2} \equiv 0$ and as $\phi_2 = -\phi_3 u_1$ we are facing two situations: $u_1 \equiv 0$ or $\phi_3 \equiv 0$. If $u_1 \equiv 0$, then the extremal is an abnormal one because (9). If $\phi_3 \equiv 0$, then from (11) we have $\dot{\phi_1} = 0$, i.e $\phi_1 = cte$. The case $\phi_1 \equiv 0$ corresponds to an abnormal extremal so assume $\phi_1 \neq 0$. From (12), we have that $u_1 \equiv \pm 1$ and we can deduce from (11) that $u_2 = \frac{\phi_4}{|\phi_1|}$ from (11).

This can be summarized in the following proposition.

Proposition 4 *Along a singular arc of extremal we have three situations:*
- $\phi_1 = \phi_2 \equiv 0$ *it is an abnormal (or a trivial singular).*
- $\phi_1 \equiv 0, \phi_2 = cte \neq 0, u_2 \equiv \pm 1$ *and* $u_1 \equiv 0$ *or* $\phi_i \equiv 0$ *for* $i = 3, 4, 5, 6$ *(in which case u_1 can have any expression) .*
- $\phi_2 \equiv 0, \phi_1 = cte \neq 0, u_1 \equiv \pm 1$ *and u_2 is determined by* $u_2 = \frac{\phi_4}{|\phi_1|}$.

Actually, for ϕ_2 singular trajectories, it is even possible to compute the expression of the corresponding closed loop control :

Proposition 5 *:*
- *The only arcs of extremal on which $\beta \equiv 0$ correspond to $u_1 \equiv \pm 1$, $u_2 \equiv 0$ and $\beta_0 = 0$. They are singular extremals for ϕ_2 (i.e. $\phi_2 \equiv 0$). In these cases, the tractor and the trailer move on the same straight line. It is clearly an optimal trajectory.*
- *On a ϕ_2 singular arc of extremal, if $\beta(t) \neq 0$ on an a non empty interval, then $u_1 \equiv \pm 1$ and*

$$u_2 = \frac{P_x \cos\theta + P_y \sin\theta - \Phi_1}{|\Phi_1| \sin\beta}$$

where P_x, P_y and $\Phi_1 \neq 0$ are constants.

proof : For a forward motion of the tractor and the trailer on the same straight line at any velocity, one can check that $(\cos(\theta_0), \sin(\theta_0), 0, 0)$ is a satisfactory adjoint vector for PMP. One can also check that in this case $\phi_2 \equiv 0$ and $\phi_1 \equiv 1$. Therefore, the only extremal where $\beta \equiv 0$ corresponds to $u2 \equiv 0$ and $u_1 \equiv 1$ (resp. $u_1 \equiv -1$) for forward (resp. backward) motions. Now, assume $\beta \neq 0$ on a non-empty interval. Recall that $\phi_1(t) = p_x(t)\cos\theta(t) + p_y(t)\sin\theta(t) - p_\beta(t)\sin\beta(t)$. In our case, $\phi_1(t) = \Phi_1 \neq 0$. In other respects, applying (4) to the Hamiltonian (8) we get $\dot{p_x}(t) = \dot{p_y}(t) = 0$ for almost all t. Therefore, $p_x(t) = P_x$ and $p_y(t) = P_y$. Eventually, a direct computation of ϕ_4 in the case of our system shows that $\phi_4(t) = p_\beta(t)$. Therefore, we have:

$$\Phi_1 = P_x \cos\theta(t) + P_y \sin\theta(t) - \phi_4(t)\sin\beta(t)$$

From lemma 4 we know that in our case $u_2 = \phi_4/|\Phi_1|$, hence the result.

5 Conclusion

This paper deals with the problem of time optimal motion planning for a mobile robot with a trailer. The main tool for our analysis of the time optimal trajectories is indeed the Pontryagin maximum principle. This principle provides a necessary condition for a trajectory to be optimal. We have to deal with two kind of extremals, the regular and the singular ones.

It is well known that the singular extremals play an important role in the optimal strategy (see [2] for example) and in this article we describe the regular and singular extremals for our problem. A first extension will be to give a complete description of these trajectories for an arbitrarily number of trailers. In an other side, the PMP does not provide enough information to characterize the optimal paths and has to be supplemented by other geometric techniques. For example it allows for a concatenations of regular and singular arcs of an arbitrarily large number N. To find a bound on the number of concatenation we have to analyze the switching functions with the help of the Lie brackets. In our future work we aim to extract the information contained in the Structure switching equations in order to provide this bound. Then the idea will be to find a general bound depending on the number of trailers.

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