**MATH 100 ♢ Survey of Mathematics ♢ Fall 2007**

## René Descartes and Analytic Geometry

### What is Analytic Geometry?

- **Ingredients:** (Euclid's) geometry, the real numbers $\mathbb{R}$
- **The tool:** A coordinate system
- **The result:** Geometry can be treated and transmitted algebraically and numerically.

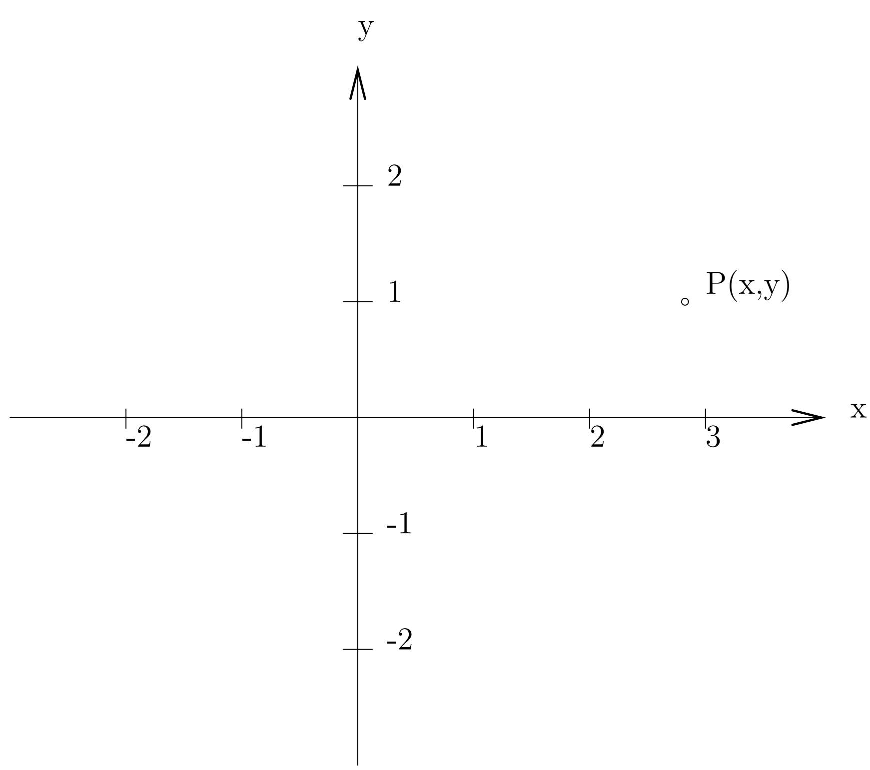



FIGURE 1. Cartesian Coordinates

### An Example using coordinates

```
\setlength{\unitlength}{0.00083333in}
\begin{picture}(10200,7971)(0,-10)
\put(683,7212){\circle{50}}
\put(983,4887){\circle{1544}}
\drawline(83,6012)(1883,6612)
\put(683,7812){\bf Geometry}}}}
\put(2500,7812){\bf Analytic Geometry}}}}
\put(2500,7212){$(x,y)$}}}
\end{picture}
```

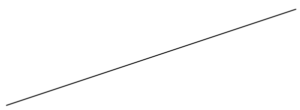
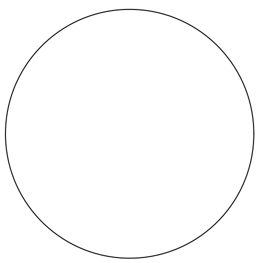
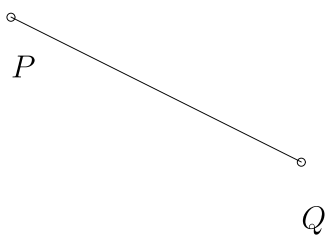
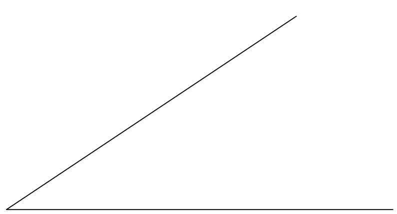
| **Geometry** | **Analytic Geometry** | **Name** |
|---|---|---|
|  | $(x, y)$ | point |
|  | $Ax + By + C = 0$ | straight line |
|  | $(x-h)^2 + (y-k)^2 = r^2$ | circle, center $(h, k)$, radius $r$ |
|   | $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ | distance $P$ to $Q$ |
|  | $m = \frac{y_2 - y_1}{x_2 - x_1}$ | slope, inclination versus x-axis |

FIGURE 2. Translation table

**In what sense does $Ax + By + C = 0$ describe a straight line?**

**Example.** A point $(a, b)$ is on the straight line $\ell$ given by $3x - 5y + 3 = 0$ if and only if $3a - 5b + 3 = 0$, i.e., a point is on the line if and only if the coordinates of the point satisfy the equation of the straight line.

$(1, 1)$ is not on the line. $(-6, -3)$ is on the line.

Hence,

$$\ell = \{(x, y) \mid 3x - 5y + 3 = 0\}$$

**Example.** A point $(a, b)$ is on the circle $\mathcal{C}$ given by $(x-5)^2 + (y+3)^2 = 25$ if and only if $(a-5)^2 + (b+3)^2 = 25$, i.e., a point is on the line if and only if the coordinates of the point satisfy the equation of the circle.

$(1, 1)$ is not on the circle, $(8, 1)$ is on the circle.

Hence,

$$\mathcal{C} = \{(x, y) \mid (x - 5)^2 + (y + 3)^2 = 25\}$$

**René Descartes (1596-1650)**

- Lesser nobility, but inheritence from mother rendered Descartes financially independent.
- Educated at a Jesuit college entering at the age of eight years, leaving at age 16.
- Being in poor health he was granted permission to remain in bed until 11 o'clock in the morning, a custom he maintained until the year of his death.
- Law degree from Poitiers in 1616.
- Enlisted as gentlemen soldier. After two years in Holland he travelled through all of Europe. In a hot room revelation of "the foundation of a marvellous science" and calling to pursue it.
- Settled down in Holland.
- First major treatise on physics, Le Monde, ou Traité de la Lumière. News that Galileo Galilei was condemned to house arrest. Decided not to risk publication. World starts with original vorteces and develops according to laws of mechanics.

- Wrote Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences. Three appendices were La Dioptrique, Les Météores, and La Géométrie. He popularizes the analytic approach to geometry.
- In 1649 Queen Christina of Sweden persuaded Descartes to come to Stockholm as her personal tutor. The Queen worked early in the morning. After only a few months in the cold northern climate, Descartes died of pneumonia.
- Descartes constitutes "the turning point between medieval and modern mathematics".

**Discours de la Méthode (Discussion of Method)**

- Clear influence of Euclid: looking for basic principles, logical deduction.
- Doubt! Disregard acquired knowledge, start from the beginning.
- "I think therefore I am".
- God exists because there must be perfection.
- Search for basic principles and derive (explain) phenomena.
- Verify the truth of the basic principles by experimentally checking the consequences.

## From Discours de la Méthode

I thought the following four [rules] would be enough, provided that I made a firm and constant resolution not to fail even once in the observance of them. The first was never to accept anything as true if I had not evident knowledge of its being so; that is, carefully to avoid precipitancy and prejudice, and to embrace in my judgment only what presented itself to my mind so clearly and distinctly that I had no occasion to doubt it. The second, to divide each problem I examined into as many parts as was feasible, and as was requisite for its better solution. The third, to direct my thoughts in an orderly way; beginning with the simplest objects, those most apt to be known, and ascending little by little, in steps as it were, to the knowledge of the most complex; and establishing an order in thought even when the objects had no natural priority one to another. And the last, to make throughout such complete enumerations and such general surveys that I might be sure of leaving nothing out. These long chains of perfectly simple and easy reasonings by means of which geometers are accustomed to carry out their most difficult demonstrations had led me to fancy that everything that can fall under human knowledge forms a similar sequence; and that so long as we avoid accepting as true what is not so, and always preserve the right order of deduction of one thing from another, there can be nothing too remote to be reached in the end, or to well hidden to be discovered.

If we possessed a thorough knowledge of all the parts of the seed of any animal (e.g. man), we could from that alone, by reasons entirely mathematical and certain, deduce the whole conformation and figure of each of its members, and, conversely if we knew several peculiarities of this conformation, we would from those deduce the nature of its seed.

It is not enough to have a good mind. The main thing is to use it well.

If you would be a real seeker after truth, you must at least once in your life doubt, as far as possible, all things.

## Optics

Descartes explains refraction, reflection, function of the eye, lenses, how to construct lenses.

## Meterology

Descartes explains the rainbow, develops a theory of the weather. “Despite its many faults, the subject of meteorology was set on course after publication of Les Météores particularly through the work of Boyle, Hooke and Halley.”

**On geometry**

- Replace the geometric algebra of the Greeks by numerical algebra. Think of line segments as numbers. Then, e.g., $a^2 + b$ makes sense.
- Many problems of geometry can be solved by working with lengths of line segments. Solves a problem of antiquity.

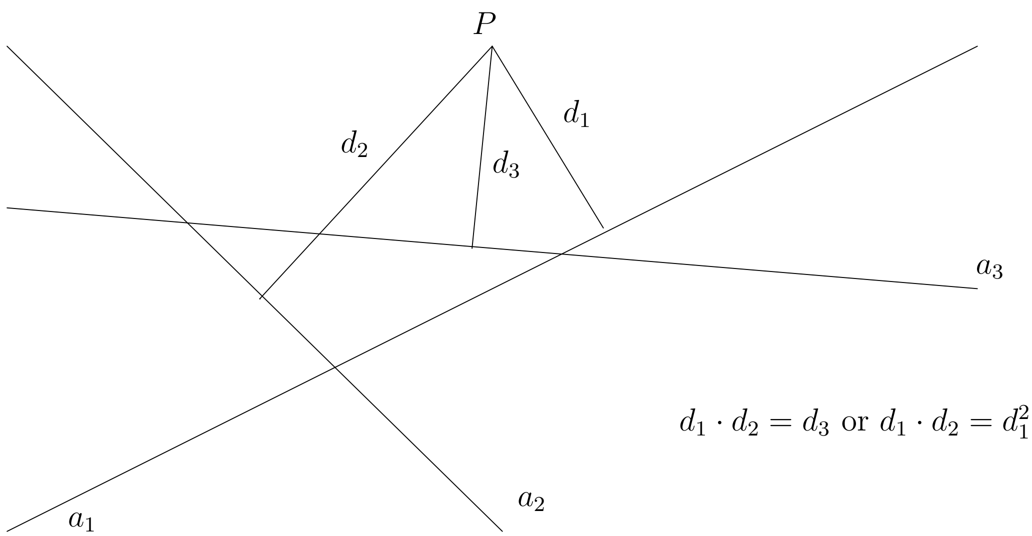



FIGURE 3. A problem of Appolonius and Pappus

- **Curves before Descartes** See http://xahlee.org
- Call the unknown line segments $x, y, z, \ldots$ and the given line segments $a, b, c, \ldots$, find relations between them by computing the same quantity in two ways. Solve the equations.

- Example: Find $x, y$ in Figure 4.

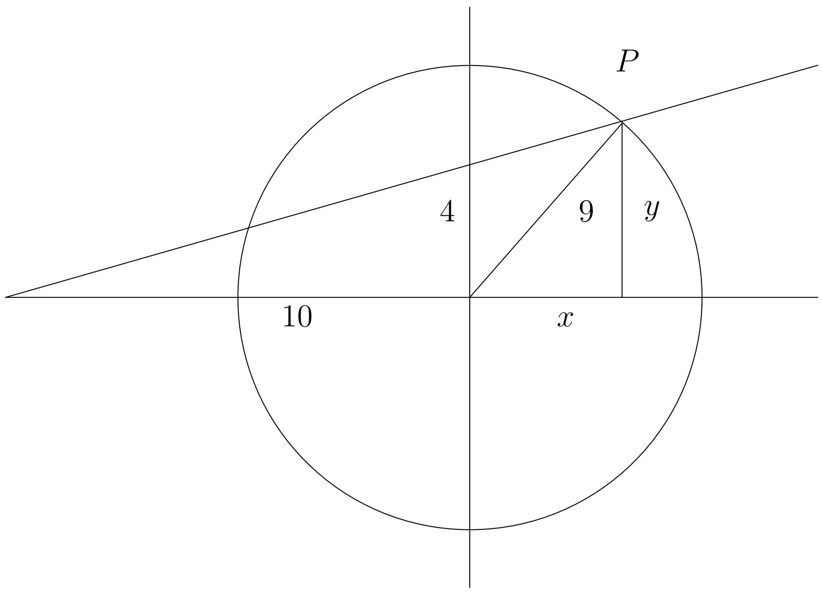



FIGURE 4. Intersecting line and circle

$$\frac{y}{x+10} = \frac{4}{10} = \frac{2}{5}, \qquad x^2 + y^2 = 9^2 = 81$$

simplifies to

$$2x - 5y + 20 = 0, \qquad x^2 + y^2 = 81$$

leads to

$$y = \frac{2}{5}x + 4, \qquad \frac{29}{25}x^2 + \frac{16}{5}x - 65 = 0.$$

## Other coordinate systems

(1) **Polar coordinates.** $r = \theta$ is the equation of the Archimedean spiral.

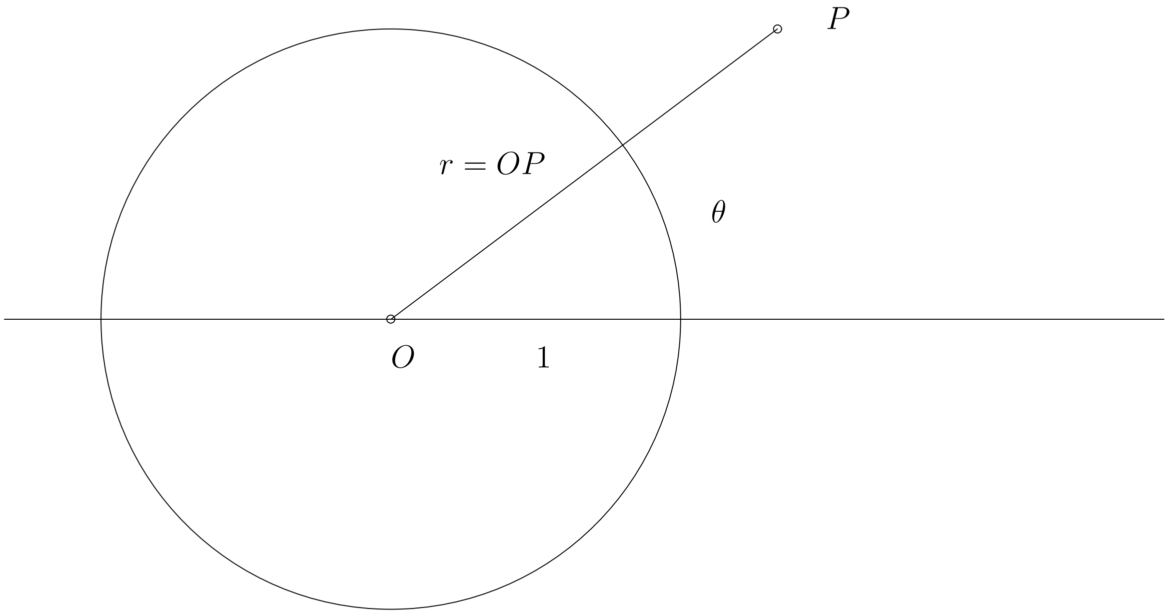



FIGURE 5. Polar coordinates

(2) **Cylindrical coordinates**

(3) **Spherical coordinates**

**General Curves**

- Curves can be described algebraically with respect to lines of reference.

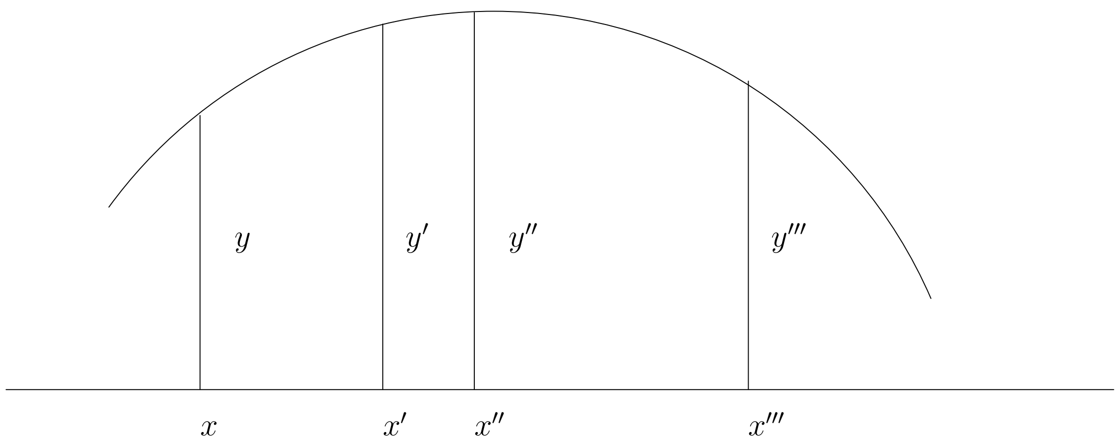



FIGURE 6. General curves according to Descartes

- Displays mechanisms that result in curves whose algebraic descriptions are more and more complicated.
- Essentially explains the graph of a function $f$: graph of $f = \{(x, f(x)) \mid x \in \mathbb{R}\}$.
- Graphs of a relation $F$: graph of $F = \{(x, y) \mid F(x, y) = 0\}$
- Develops a method for finding the tangent to a curve.

## The question of dimension

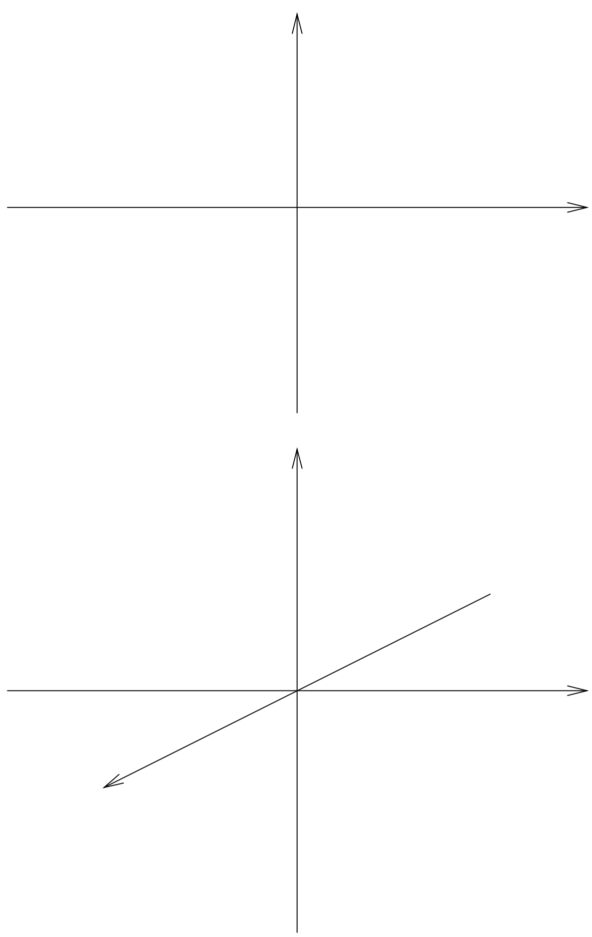



FIGURE 7

**Visualization fails but algebraically we can go on**

- $(x, y, z, t)$, $Ax+By+Cz+Dt+E=0$, $Ax^2+By^2+Cz^2+Dt^2+E=0$
- $(x, y, z, t, u)$, $Ax+By+Cz+Dt+Eu+F=0$, $Ax^2+By^2+Cz^2+Dt^2+Eu^2+F=0$
- $(x_1, x_2, \ldots, x_n)$, $A_1x_1 + A_2x_2 + \cdots + A_nx_n + B = 0$, $A_1x_1^2 + A_2x_2^2 + \cdots + A_nx_n^2 + B = 0$