MORE IMPORTANT THEOREMS AND DEFINITIONS

1. Inverse Functions and Examples

Definition 1. *A function $f : A \to B$ is said to be one-to-one if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.*

Definition 2. *A function $f : A \to B$ is said to be onto if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$.*

Theorem 1 (Horizontal Line Test)**.** *A function f is one-to-one if and only if every line parallel to the $x-axis$ intersects the graph of f in at most one point.*

Definition 3. *Let $f : A \to B$ and $g : B \to A$ be functions. If $f(g(b)) = b$ and $g(f(a)) = a$ for all $a \in A$ and $b \in B$, then f is said to be invertible and g is said to be the inverse of f. If f is invertible, the inverse of f is often written as $f^{-1}(x)$.*

Theorem 2. *A function $f : A \to B$ is invertible if and only if f is one to one and onto.*

Theorem 3. *If $f : A \to B$ is invertible, the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.*

Theorem 4. *If $f : A \to B$ is invertible and continuous, then it's inverse is continuous.*

Theorem 5 (Inverse Function Theorem for Single Variable Real Calculus)**.** *If f is invertible and differentiable with inverse function g and $f'(g(a)) \neq 0$, then g is differentiable and*

$$g'(a) = \frac{1}{f'(g(a))}$$

Definition 4. *Let $a > 0$ be a real number. An exponential function is a function of the form $f(x) = a^x$. The value of a^x is given by the following:*

1. *If $x \in \mathbb{N}$, then $a^x = \underbrace{a \cdot a \cdot a \cdots a}_{x-times}$.*
2. *If $x = 0$, then $a^x = 1$.*
3. *If $x \in \mathbb{Z}$ and $x < 0$, then $a^x = 1/a^{|x|}$.*
4. *If $x = p/q$ is rational, $q > 0$ then $a^x = a^{p/q} = \sqrt[q]{a^p}$. This definition is independent of the rational decomposition of x.*
5. *If x is irrational and $\{r_n\}$ is a sequence of rational numbers converging to x, then $a^x = \lim_{n\to\infty} a^{r_n}$. This definition is independent of the choice of the sequence $\{r_n\}$.*

Theorem 6 (Properties of Exponential Functions)**.** *If $a > 0$ is a real number and $f(x) = a^x$ is the exponential function defined above, then f is continuous. The domain of f is all reals and if $a \neq 1$, the range of f is $(0, \infty)$. If $a = 1$, the range of f is $\{1\}$.*

(1) *If $0 < a < 1$, then f is decreasing and moreover, $\lim_{x \to -\infty} a^x = \infty$ and $\lim_{x \to \infty} a^x = 0$.*
(2) *If $a > 1$, then f is increasing and moreover, $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to \infty} a^x = \infty$.*

Theorem 7 (Laws of Exponents)**.** *Let $a, b > 0$ and $x, y \in \mathbb{R}$. Then:*

(1) $a^x a^y = a^{x+y}$
(2) $a^x / a^y = a^{x-y}$
(3) $(a^x)^y = a^{xy}$
(4) $(ab)^x = a^x b^x$

Definition 5 (The Natural Logarithm)**.** *The natural logarithm is defined by an integral as follows:*

$$ln(x) = \int_1^x \frac{1}{t}\, dt$$

The notation for the natural logarithm, ln is taken from the French, logarithme naturelle.

Theorem 8 (Properties of the Natural Logarithm)**.** *The natural logarithm function is continuous and differentiable on $(0, \infty)$ and has the following properties:*

(1) $ln(1) = 0$.
(2) *$ln(x)$ is strictly increasing on $(0, \infty)$ and hence is one-to-one.*
(3) *The range of $ln(x)$ is $(-\infty, \infty)$.*
(4) $\frac{d}{dx} ln(x) = \frac{1}{x}$.
(5) *If $a > 0$ and $b \in \mathbb{R}$, $ln(a^b) = b \cdot ln(a)$.*
(6) *If $a, b > 0$, $ln(ab) = ln(a) + ln(b)$.*
(7) *If $a, b > 0$, $ln(a/b) = ln(a) - ln(b)$*

Definition 6 (Definition of the number e)**.** *The number e is the unique positive real number such that $ln(e) = 1$. A numerical approximation to e accurate to 20 decimal places is $e \approx 2.71828182845904523536$. The number e is often called Euler's constant.*

Theorem 9 (Properties of e and e^x)**.**

(1) *e is an irrational real number.*
(2) *$e^a = b$ if and only if $ln(b) = a$.*
(3) *e^x is differentiable on all of $\mathbb{R}$ and $\frac{d}{dx} e^x = e^x$.*

Theorem 10. *The following definitions of the number e are all equivalent.*

(1) *e is the unique positive real number such that $ln(e) = 1$.*
(2) $e = \lim_{n \to \infty} (1 + 1/n)^n$

(3) *e is the unique number that satisfies the following limit equation:*

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h}$$

Definition 7. *Let $a > 0$, $a \neq 1$ be a real number. The logarithmic function with base a is the inverse function of the one-to-one function a^x. It is denoted by the symbol $\log_a(x)$.*

Theorem 11 (Properties of General Logarithmic Functions)**.** *Let $a > 1$ be a real number. The function $f(x) = \log_a(x)$ is continuous on $(0, \infty)$ and has range $(-\infty, \infty)$. The following properties hold:*

(1) $\log_a(x)$ *is increasing,* $\lim_{x\to\infty} \log_a(x) = \infty$*, and* $\lim_{x\to 0^+} \log_a(x) = -\infty$.
(2) $\log_a(1) = 0$
(3) *If* $x, y > 0$, $\log_a(xy) = \log_a(x) + \log_a(y)$.
(4) *If* $x, y > 0$, $\log_a(x/y) = \log_a(x) - \log_a(y)$.
(5) *If* $x, y > 0$, $\log_a(x^y) = y \log_a(x)$.
(6) *For any* $a > 0$, $a \neq 1$, *we have*

$$\log_a(x) = \frac{ln(x)}{ln(a)}$$

Theorem 12. *Let $a > 0$. The derivatives of exponential and logarithmic functions are summarized in the following:*

(1) $\frac{d}{dx} ln|x| = \frac{1}{x}$.
(2) $\frac{d}{dx} a^x = (ln(a))a^x$.
(3) $\frac{d}{dx} \log_a(x) = \frac{1}{xln(a)}$.

Theorem 13 (The Power Rule)**.** *Using logarithmic differentiation we can finally completely establish the Power Rule. Let n be any real number and define $f(x) = x^n$. Then f is differentiable on any open interval where it is defined and $f'(x) = nx^{n-1}$.*

Definition 8 (Inverse Trigonometric Functions)**.** *The inverse trigonometric functions, with their standard domains, are defined below:*

(1) *The function $arcsin(x)$ is the inverse function of $sin(x)$ on $[-\pi/2, \pi/2]$. It has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.*
(2) *The function $arccos(x)$ is the inverse function of $cos(x)$ on $[0, \pi]$. It has domain $[-1, 1]$ and range $[0, \pi]$.*
(3) *The function $arctan(x)$ is the inverse function of $sin(x)$ on $(-\pi/2, \pi/2)$. It has domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$.*
(4) *The function $arccsc(x)$ is the inverse function of $csc(x)$ on $(0, \pi/2] \cup (\pi, 3\pi/2]$. It has domain $(-\infty, -1] \cup [1, \infty)$ and range $(0, \pi/2] \cup (\pi, 3\pi/2]$.*
(5) *The function $arcsec(x)$ is the inverse function of $sec(x)$ on $[0, \pi/2) \cup [\pi, 3\pi/2)$. It has domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi/2) \cup [\pi, 3\pi/2)$.*

(6) *The function* $arccot(x)$ *is the inverse function of* $cot(x)$ *on* $(0, \pi)$*. It has domain* $(-\infty, \infty)$ *and range* $(0, \pi)$*.*

Theorem 14. *The six inverse trigonometric functions are differentible and their derivatives are given by:*

(1) $\frac{d}{dx} arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
(2) $\frac{d}{dx} arccos(x) = -\frac{1}{\sqrt{1-x^2}}$
(3) $\frac{d}{dx} arctan(x) = \frac{1}{1+x^2}$
(4) $\frac{d}{dx} arccsc(x) = -\frac{1}{x\sqrt{x^2-1}}$
(5) $\frac{d}{dx} arcsec(x) = \frac{1}{x\sqrt{x^2-1}}$
(6) $\frac{d}{dx} arccot(x) = -\frac{1}{1+x^2}$

Definition 9 (Indeterminant Forms)**.** *Let* $f(x)$ *and* $g(x)$ *be continuous functions. Consider the symbol*

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

(1) *If* $f(x) \to 0$ *and* $g(x) \to 0$ *as* $x \to a$*, then the symbol is called an indeterminant form of type* $\frac{0}{0}$*.*
(2) *If* $f(x) \to \pm\infty$ *and* $g(x) \to \pm\infty$ *as* $x \to a$*, then the symbol is called an indeterminant form of type* $\frac{\infty}{\infty}$*.*

Theorem 15 (L'Hôpital's Rule)**.** *Suppose that* $f(x)$ *and* $g(x)$ *are differentible on an open interval* I *containing* a*,* $g'(x) \neq 0$ *on* $I \setminus \{a\}$ *and that* $\lim_{x \to a} \frac{f(x)}{g(x)}$ *is an indeterminate form of type* $0/0$ *or* ∞/∞*. Then:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists or is $\pm\infty$*. The theorem remains true if we replace* $x \to a$ *with* $x \to a^+$*,* $x \to a^-$*,* $x \to \infty$*, or* $x \to -\infty$*.*

Strategy 1 (Other Indeterminant Forms)**.** *Some expressions can be written in such a way that l'Hôpitals Rule can be applied even though they are not initially in the form of one of the indtermnate forms* $0/0$ *or* ∞/∞*. The associated indeterminate forms and the strategies associated to them are given below.*

(1) *If* $\lim_{x \to a} f(x) = 0$ *and* $\lim_{x \to a} g(x) = \pm\infty$*, the symbol* $\lim_{x \to a} f(x)g(x)$ *is an indeterminant form of type* $0 \cdot \infty$*. The strategy is to write* $fg = \frac{f}{1/g}$ *or* $fg = \frac{g}{1/f}$*, which will have an indeterminant form to which l'Hôpital's Rule might apply.*
(2) *If* $\lim_{x \to a} f(x) = \infty$ *and* $\lim_{x \to a} g(x) = \infty$*, the symbol* $\lim_{x \to a} f(x) - g(x)$ *is an indeterminant form of type* $\infty - \infty$*. The strategy is to write* $f - g$ *as a quotient or a product which will have an indeterminant form to which l'Hôpital's Rule might apply.*

(3) *If* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, *the symbol* $\lim_{x\to a} f(x)^{g(x)}$ *is an indeterminant form of type* 0^0. *The strategy is to either look at the limit of the natural logarithm of the expression or to rewrite the expression as an exponential* $f(x)^{g(x)} = e^{g(x)ln(f(x))}$. *The result will have an indeterminant form to which l'Hôpital's Rule might apply. The original limit can then be determined using the continuity of the* e^x *or* $ln(x)$ *appropriately.*

(4) *If* $\lim_{x\to a} f(x) = \infty$ *and* $\lim_{x\to a} g(x) = 0$, *the symbol* $\lim_{x\to a} f(x)^{g(x)}$ *is an indeterminant form of type* ∞^0. *The strategy is the same as with the indeterminant form* 0^0.

(5) *If* $\lim_{x\to a} f(x) = 1$ *and* $\lim_{x\to a} g(x) = \infty$, *the symbol* $\lim_{x\to a} f(x)^{g(x)}$ *is an indeterminant form of type* 1^∞. *The strategy is the same as with the indeterminant form* 0^0.

2. Integration Techniques and Numerical Integration

Theorem 16 (Integration by Parts Formula)**.** *Let* $f(x)$ *and* $g(x)$ *be differentiable functions on a closed interval* $[a, b]$. *We have the following integration by parts formula for indefinite and definite integrals*

(1)
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

(2)
$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)\,dx$$

Strategy 2 (ILATE)**.** *To pick the* u *in an integration by parts problem, it often works to choose* u *to be the function that shows up first in the ILATE list:*
I=Inverse trigonometric functions
L=Logarithmic functions
A=Algebraic functions
T=Trigonometric functions
E=Exponential functions

Strategy 3 (Trigonometric Integrals)**.** *To evaluate products of powers of trigonometric functions, use one of the following startegies:*

(1) $\int sin^m(x)cos^n(x)\,dx$

  (a) *If* n *or* m *is odd, write the function* $f(x)$ *with this odd power* $2k+1$ *as* $(f(x))^{2k+1} = f(x)((f(x))^2)^k$. *Then substitute out* $(f(x))^2$ *using the formula* $cos^2(x) + sin^2(x) = 1$. *The resulting integral can now be performed using u-substitution with* u *whichever of sine and cosine that* f *is not.*

  (b) *If* n *and* m *are both even, apply the following half-angle formulas:*
$$sin^2(x) = \frac{1}{2}(1 - cos(2x))$$

$$cos^2(x) = \frac{1}{2}(1 + cos(2x))$$

(2) $\int tan^m(x) sec^n(x)\, dx$
   (a) *If* $n = 2k$ *is even* ...
   (b) *If* $m = 2k + 1$ *is odd* ...

Theorem 17. *The integrals of all of the trigonometric functions are as follows*

(1) $\int sin(x)\, dx = -cos(x) + C$
(2) $\int cos(x)\, dx = sin(x) + C$
(3) $\int tan(x)\, dx = ln|sec(x)| + C$
(4) $\int csc(x)\, dx = ln|csc(x) - cot(x)| + C$
(5) $\int sec(x)\, dx = ln|sec(x) + tan(x)| + C$
(6) $\int cot(x)\, dx = ln|sin(x)| + C$

3. Sequences and Series

4. Power Series and Representations of Functions

5. Differential Equations and Applications