THE BIG IDEA:
A CALCULUS TEXT

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1. Equality and Approximation

1.1. **The Analytic Mode of Equality.** What does it mean for two things to be equal to each other? Certainly the number 1 is the same as the number 1. This is an example of a tautology and is not very interesting. Consider the symbols $\frac{1}{2}$ and $\frac{2}{4}$. These two symbols look very different but represent the same number. Why? A typical and not entirely incorrect explanation would be that they both represent the same decimal number .5. We divide 2 into 1 and 4 into 2 and we see that in both instances we get .5. Thus $\frac{1}{2}$ equals $\frac{2}{4}$.

This procedure is unsatisfying for two reasons. First of all, the only thing we have really proved is that .5 is another representation of each of the fractions. We then use the tautology that .5 and .5 are the same to conclude that the fractions are equal. What does one do if he has two different decimal representations of the same number? This brings us to the second point. It is true that two decimal representations are equal if they are equal in every digit. However, it is not true that just becuse two numbers have different decimal representations that they must be different numbers. For example, take the following often disbelieved fact.

Theorem 1. $1 = .999\ldots$

Proof. Let $a = .999\ldots$. Then $10a - 9 = a$. Solving for a gives us that $a = 1$. □

Many people express shock when presented with this information. They wonder why there isn't some other number "between" 1 and $.999\ldots$. The belief is that 1 and $.999\ldots$ are close together but that the distance between them is positive. The above proof should convince you of the theorem's verity, but it is instructive to dissect the false intuition.

Suppose that $1 - .999\ldots > 0$. Let $x = 1 - .999\ldots$. The number x has a decimal expansion $.a_1a_2a_3\ldots$. Let a_k be the first nonzero digit. Since $a_k \geq 1$, $(.999\ldots)+(.000\ldots0a_k0\ldots) > 1$. Then we must have that $.999\ldots+x > 1$. This is preposterous by definition of x. Thus, $a_i = 0$ for all i and we conclude that $x = 0$. So there really cannot be a number between 1 and $.999\ldots$!

Theorem 2. *The Analytic Mode of Equality. Two real numbers x and y are equal if and only if for every $\epsilon > 0$, $|x - y| < \epsilon$. In other words, x and*

y are equal if and only if the distance between x and y is smaller than every posive number.

Proof. If $x = y$, then $|x - y| = 0$. Thus $|x - y|$ is less than every positive number. Now suppose that $0 \leq |x - y| <$ every positive number. If $x \neq y$ then $|x - y| \neq 0$. Let $a = |x - y|$. Then $0 < \frac{a}{2} < |x - y|$. This contradicts our hypothesis, so we must have that $x = y$. □

This theorem gives us an effective way of determining when two numbers are the same. It may not seem that impressive at this point since we have only applied it in the case of $.999\ldots = 1$. We even have a proof of this fact that does not use the Mode of Equality at all. The power of the theorem shows up when we start asking questions about numbers like $\sqrt{2}, \pi$, and e. Numbers like these do not have repeating decimal expansions. Shortly, we will look at $\sqrt{2}$ and use a polynomial equation to approximate its value. This is essentially how we first proved that $.999\ldots = 1$. We used the decimal $.999\ldots$ to set up the linear equation $10a - 9 = a$ in the variable a and then solved for a. For $a = \sqrt{2}$, we could use trial and error on the equation $a^2 - 2 = 0$ to get a decimal expansion close to $\sqrt{2}$. The numbers e and π, however, have no useful algebraic equations for which they are roots. Numbers with this property are called transcendental numbers. Hermite proved that e is transcendental in 1873 and Lindemann proved that π is transcendental in 1882. Essentially the only way to find information about these numbers is to use analytic techniques. We will consider e and π in more detail later.

1.2. **Equivalence Relations.** We've seen that we have to be careful about how we determine when two real numbers are equal. It may seem an unnatural discussion, but it is a mathematically important one. In fact many different notions of equality exist when considering different systems of objects. The most commonly used notion of equality that differs wildly from that of the real numbers is "clock arithmetic". Here the numbers are the hours on a standard 12 hour clock, $\{12, 1, 2, 3, \ldots, 11\}$. We will say that two numbers are "equal" if they correspond to the same 12 hour time. To avoid confusion we will replace the standard symbol of equality for the reals with the symbol $\sim$. For example, $8 + 5 \sim 1$ loosely corresponds to the statement "five hours after 8 o'clock is 1 o'clock". Also, adding 12 hours to any given time returns the original time. Hence $x + 12 \sim x$ for all hours x.

As far as a mathematician is concerned an equality, or equivalence relation, need only have three properties.

Definition 1. *An* **equivalence relation**, $\sim$, *is any relation on a set A that satisfies the following three axioms.*

1. *$a \sim a$ for all $a \in A$. (Reflexive Property)*
2. *If $a \sim b$ then $b \sim a$.(Symmetric Property)*
3. *If $a \sim b$ and $b \sim c$ then $a \sim c$. (Transitive Property)*

Given a set A and an equivalence relation $\sim$, if $a \sim b$, we say that a and b are in the same equivalence class (or just class for short).

Let's see why the relation $\sim$ on the clock numbers gives rise to an equivalence relation. We'll need to be a little more precise about how we defined our relation. Let the set of integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, represent the hours measured forwards and backwards from a given point. In clock time, two hours are the same if they differ from each other by $\pm 12, \pm 24, \pm 36$, or $12k$ hours (k any integer). So if x and y are integers, $x \sim y$ if and only if $x - y = 12k$ for some integer k.

Verifying the axioms of an equivalence relation for clock time now just comes down to manipulating the equation derived above.

- (Reflexive Property) Since $x - x = 0 = 12 \cdot 0$ for all integers x, the reflexive property holds.
- (Symmetric Property) Suppose $a \sim b$. Then $a - b = 12k$ for some integer k. Multiplying the equation by -1 gives $b - a = 12 \cdot (-k)$ and thus the symmetric property holds.
- (Transitive Property) Finally, if $a \sim b$ and $b \sim c$, then $a - b = 12k$ and $b - c = 12l$ for some integers l and k. Thus, $a - c = (a - b) + (b - c) = 12(k + l)$ and the transitive property holds.

The Mode of Equality defines an equivalence relation on the set of all real numbers. The proof is an exercise in the manipulation of absolute values, with the transitive property being the most difficult to verify. By Theorem 1, we know that 1 and $.999\dots$ are in the same class. Previously we said that 1 and $.999\dots$ are representations of the same number. This is true in the sense that the class of things that are equivalent to $.999\dots$ by the Mode of Equality is the same as the class of things that are equivalent to 1 by the Mode of Equality. Other elements of this equivalence class include $\frac{2}{2}$ and $\frac{\pi}{\pi}$.

Of course, the equivalence relation is the same as the familiar $=$ for the real numbers. To verify that the familiar $=$ is an equivalence relation is far more difficult because that requires us to answer the question "What is a real number?". This question was answered independently by Dedekind and Cauchy. Here we will briefly discuss the method of Cauchy. His basic idea was that a real number can best be described by a sequence of rational numbers. For example the sequence $< 3, 3.1, 3.14, 3.141, 3.1415, \dots >$ is a sequence of rational numbers with property that as you go farther out, the elements of the sequence get closer and closer to π. This sequence is said to converge to π. In general, there are many sequences that converge to a given real number. We define the equivalence relation $=$ on the set of all convergent sequences by saying that two sequences are equal if their term by term difference gets arbitrarily close to 0. The classes under this equivalence relation are the real numbers. The details of this construction are quite complex and well beyond the scope of these notes. This outline, however, emphasizes the important role that the rational numbers play in analysis and calculus.

1.3. **Rational Numbers.** Consider the set of all symbols of the form p/q where p and q are integers and $q \neq 0$. An equivalence relation $\sim$ can be defined on this set as follows: $p/q \sim a/b$ if and only if $pb = aq$. Let's verify that this is an equivalence relation.

- (Reflexive Property) Since $pq = pq$, the refelexive property is satisfied.
- (Symmetric Property) The symmetric property just follows from the fact that $=$ for the integers is an equivalence relation.
- (Transitive Property) Now suppose that $p/q \sim a/b$ and $a/b \sim c/d$. To verify the transitive property, we must show that $p/q \sim c/d$. By hypothesis, (1) $pb = aq$ and (2) $ad = bc$. Multiply (1) by d. Then we have $pdb = pbd = aqd = qad = qbc = qcb$. It is the first and last parts in the string of equalities that interest us. Our symbols were defined so that the denominator is not zero. Thus $b \neq 0$ and we can divide the equation $pdb = qcb$ by b in order to get the desired $pd = qc$.

This relation is the familiar notion of equivalence for the rational numbers. It is also an equivalence relation that agrees with the Analytic Mode of Equality when we assume that x and y are rational. Suppose that $p/q \sim a/b$. Then $pb = aq$ and for all $\epsilon > 0$:

$$\left|\frac{p}{q} - \frac{a}{b}\right| = \left|\frac{pb-aq}{qb}\right| = 0 < \epsilon$$

Thus, since $\frac{1}{2}$ and $\frac{2}{4}$ are in the same class under $\sim$, they are in the same class under the Analytic mode of Equality. It is also true that if x and y are rational and equivalent under the Analytic Mode of Equality then $x \sim y$.

A number that fails placement in the format p/q is called an irrational number. A simple test that one can use to determine whether or not a number is rational is to find the decimal expansion of the rational number.

Theorem 3. *A number is rational if and only if its decimal expansion is finite or repeating.*

Sketch of Proof: Suppose that $x = p/q$ is rational. Let $\{r_1, r_2, r_3, ...\}$ be list of remainders obtained from dividing q into p in their proper order. By the division algorithm, each of the remainders is taken to be smaller than q. So the number of different r_i is q. If $r_i = 0$ for some i, then the list is finite and p/q has finite decimal expansion. Otherwise, since there are but $q + 1$ different r_i and an infinite list to fill, we must have $r_j = r_k$ for some $k > j$. Then $r_{j+1} = r_{k+1}$ and the digits in the quotient repeat.

Now suppose, without loss of generality, that $0 < x < 1$. If x has n digits in its decimal expansion, then $10^n x$ is an integer and x is rational. Suppose that x has a repeating decimal expansion and that the repeating part is of length n starting at the k-th digit. Then $x = .a_1 \ldots a_{k-1} b_1 \ldots b_n b_1 \ldots b_n \ldots$. Then x satisfies the equation:

$$10^{k-1}x - a_1a_2\dots a_{k-1} = \frac{b_1\dots b_n}{10^n - 1}$$

Solving for x gives x in rational form. □

From this it follows that the number .01001000100001 . . . is not a rational number. Oddly enough, most numbers are not rational. By "most" here we mean that if one were to pick a point on a number line, one would pick an irrational number with probability 1.

1.4. **Irrational Numbers and Rational Approximation.** Even though most real numbers are not rational, every real number x is arbitrarily close to a rational number in the following sense.

Theorem 4. *Let $\epsilon > 0$ be given and x a real number. Then there is a rational number y such that $|x - y| < \epsilon$. In fact, there are infinitely many rational numbers y that satisfy the inequality $|x - y| < \epsilon$*

Proof. Since every real number may be written as an integer plus a number less than 1, we can assume without loss of generality that $0 < x < 1$. Then $x = .a_1a_2a_3\dots$. Let $\epsilon > 0$ be given. There is an n so large that $\frac{1}{10^n} < \epsilon$. Let $y = .a_1a_2\dots a_n$. Then

$$|x - y| < .000\dots a_{n+1}a_{n+2}\dots \le \frac{1}{10^n} < \epsilon$$

□

Definition 2. *Let x be a real number and $\epsilon > 0$. A* **rational approximation to x with error less than** ϵ *is a rational number y such that $|x - y| < \epsilon$. Equivalently, we must have y contained in the interval $(x - \epsilon, x + \epsilon)$ for y to be a rational approximation to x with error less than ϵ.*

For example, $\pm\frac{1}{4}, \pm\frac{1}{3}, \pm\dots\frac{1}{n}$ (n a nonzero integer such that $|n| > 2$), are all rational approximations to 0 with error less than $\frac{1}{2}$. Any number x is a rational approximation to 0 with error less than $|2x|$. A good approximation will be one in which ϵ can be taken very small, like $\epsilon = .01, .001, .001$. The smaller the ϵ, the better the approximation. Of course, the best rational approximation to 0 is 0 since 0 is a rational number. We typically find rational approximations for irrational numbers. Rational mumbers don't need rational approximation because any decimal expansion for any rational number can be completely written out in a finite amount of time. This is just what Theorem 3 said. Moreover, Theorem 3 says, that in general, the decimal digits of the irrational numbers cannot be specified in a finite amount of time. Thus, if we want to know where a particular irrational number is on the number line it is necessary that we approximate it.

As an example of this idea, we give an algorithm for finding a rational approximation to $\sqrt{2}$ with any specified error. First we'll establish some useful facts.

Theorem 5. $\sqrt{2}$ *is irrational.*

Proof. Suppose that $\sqrt{2} = \frac{p}{q}$ for some integers p and q. We'll assume that the representation is chosen so that p and q have no common factors. Now, $2q^2 = p^2$. This means that 2 divides p^2 and thus 2 divides p. Then 4 divides p^2 and thus 4 divides $2q^2$. But this means that 2 divides q^2 and 2 divides q! This is a contradiction because p and q were chosen as to have no common factors. □

To find rational numbers that are close to $\sqrt{2}$, we will find integer solutions to the equation

$$\frac{p}{q} = \sqrt{2 \pm \frac{1}{q^2}}$$

As q gets larger and larger, the associated fraction will get closer and closer to $\sqrt{2}$. If we square both sides, we can put the above equation into the form:

$$p^2 - 2q^2 = \pm 1.$$

This equation is a case of Pell's equation $p^2 - nq^2 = 1$ and is known to always have integer solutions p and q for positive n that are not perfect squares. For these n, p/q is a rational approximation to $\sqrt{n}$. The case that we are considering has solutions $p = q = 1$. But 1 is not a good rational approximation to $\sqrt{2}$ since $1^2 = 1$. Choosing $p = 3$ and $q = 2$ is another solution which gives the rational approximation $3/2 = 1.5$. This is a little better since $(3/2)^2 = 9/4 = 2.25$. To find better rational approximations, we will have to find larger values of q. To do this, we note that if p_1 and q_1 are solutions, then $p_2 = p_1 + 2q_1$ and $q_2 = p_1 + q_1$ is also a solution:

$$\begin{aligned}
(p_1 + 2q_1)^2 - 2(p_1 + q_1)^2 &= p_1^2 + 4p_1q_1 + 4q_1^2 - 2(p_1^2 + 2p_1q_1 + q_1^2) \\
&= -(p_1^2 + 2q_1^2) \\
&= \pm 1
\end{aligned}$$

Using the initial solution of $p_1 = q_1 = 1$ we can create the above table of rational approximations to $\sqrt{2}$. Included are the decimal expansions for those rational numbers and the length of the repeating parts of their decimal expansions.

Of coures, we haven't bothered yet to determine the most important fact. How close are the above numbers to the actual value of $\sqrt{2}$? This is remedied in the following theorem which also sumarizes the above results.

Theorem 6. *If the positive integers p and q are solutions to the equation $p^2 - 2q^2 = \pm 1$, then p/q is a rational approximation to $\sqrt{2}$ with error less than $\frac{1}{2q^2}$. Also, $x = p + 2q$ and $y = p + q$ are solutions to $p^2 - 2q^2 = \pm 1$.*

Proof. Suppose that p and q are positive integer solutions to the equation. Then

Rational Approx.	Decimal Expansion	Length of Repeating Part
1	1	1
$\frac{3}{2}$	1.5	1
$\frac{7}{5}$	1.4	1
$\frac{17}{12}$	$1.41\overline{6}$	1
$\frac{41}{29}$	$1.4137931034\ldots$	28
$\frac{99}{70}$	$1.4\overline{142857}$	6
$\frac{239}{169}$	$1.4142011834\ldots$	78
$\frac{577}{408}$	$1.414\overline{2156862745098039}$	16
$\frac{1393}{985}$	$1.4142131979\ldots$	98
$\frac{3363}{2378}$	$1.4142131979\ldots$	140
$\frac{8119}{5741}$	$1.4142135516\ldots$	5740

$$\begin{aligned}
\left|\sqrt{2}-\tfrac{p}{q}\right| &= \left|\sqrt{2}-\sqrt{2\pm\tfrac{1}{q^2}}\right| \\
&= \left|\sqrt{2}-\sqrt{2\pm\tfrac{1}{q^2}}\right| \frac{\left|\sqrt{2}+\sqrt{2\pm\frac{1}{q^2}}\right|}{\left|\sqrt{2}+\sqrt{2\pm\frac{1}{q^2}}\right|} \\
&= \frac{1/q^2}{\left|\sqrt{2}+\sqrt{2\pm\frac{1}{q^2}}\right|} \\
&< \frac{1}{2q^2}
\end{aligned}$$

The last inequality holds because $\sqrt{2} > 1$ and $\sqrt{2\pm\frac{1}{q^2}} \geq 1$ when $q \geq 1$. □

1.5. **Exercises.**

(1) Make a list of all the decimal expansions of 0 that are equivalent under the Analytic Mode of Equality.
(2) Prove that every irrational number has a unique decimal expansion.
(3) Prove that the Analytic Mode of Equality is indeed an equivalence relation (Hint: You will need to use the Triangle Inequality to verify the transitive property.)
(4) Show that the relation $\sim$ defined defined on the set of all symbols of the form $.a_1a_2a_3\ldots$ by saying that $a_na_{n-1}\ldots a_1a_0.a_{-1}a_{-2}a_{-3}\cdots \sim b_mb_{m-1}\ldots b_1b_0.b_{-1}b_{-2}b_{-3}\ldots$ if $m=n$ and $a_i=b_i$ for all i is an equivalence relation. Is this the same as the equivalence relation induced on the reals by the Analytic Mode of Equality?

(5) Prove that the length of the repeating part of the decimal expansion of p/q is less than or equal to q.

(6) Find an upper bound on the error for all 10 of the rational approximations to $\sqrt{2}$ given in the chapter.

(7) (a) Show that if p_1 and q_1 are solutions to the equation $p^2 - 3q^2 = \pm 1$, then $p_2 = 2p_1 + 3q_1$ and $q_2 = p_1 + 2q_1$ are also solutions to the equation.

(b) Find an initial solution to the equation $p^2 - 3q^2 = \pm 1$ and use the first part of the question to find 10 rational approximations to $\sqrt{3}$.

(c) Show that the error in any rational approximation $\frac{p}{q}$ to $\sqrt{3}$ using Pell's equation is less than $\frac{1}{2q^2}$.

(d) Find the upper bound on the error for each of your 10 rational approximations to $\sqrt{3}$.